

# Entropy and Expansive Map

Elmar Lasa(elas@clermson.edu)

Our references include Michael Brin's Introduction to Dynamical System ([2]), James Brown Ergodic Theory and Topological dynamics [3] and Goodman's article [1].

## 1 Positioning the problem

Let  $(X, d)$  be a compact metric space. A homeomorphism  $f : X \rightarrow X$  is *expansive* if there exists  $\delta_0 > 0$  such that for any two distinct points  $x \neq y$  and for some  $n \in \mathbb{Z}$ ,  $d(f^n(x), f^n(y)) \geq \delta_0$ . Given a dynamical system, there are two notions of entropy: topological entropy and measure-theoretic entropy also called (we shall call) *metric entropy*. It is well known (cf.[1]) that for Hausdorff topological space, therefore for all metric spaces, the metric entropy is less than the topological entropy. Using variational principle, it is also shown that the topological entropy of a homeomorphism on a compact metric space is the supremum of its metric entropies over all Borel probability measures. We shall prove in this work that this supremum is actually attained for expansive maps. In this case,  $f$  is said to have a *measure of maximal entropy*.

## Metric Entropy

$(X, \mathcal{B}, \mu)$  is Borel space such that  $\mu[X] = 1$ .

### 1.1 Entropy for a partition

Let  $\zeta = \{C_i : 1 \leq i \leq m\}$  for  $m \in \mathbb{N}$  be a partition for  $X$  of essentially disjoint subsets of  $X$ , that is for all  $i \neq j$ , we have  $\mu[C_i \cap C_j] = 0$ . We have to set up some vocabularies[2].

- *Refinement*. The partition  $\zeta'$  is said to be a refinement of  $\zeta$ , denoted  $\zeta' \geq \zeta$  if  $\forall C \in \zeta \exists C' \in \zeta' (C' \subseteq C \pmod{0})$ , that is every element in  $\zeta$  is essentially contained in an element in  $\zeta'$ ;
- *Equivalence*. Two partitions are said to be equivalent if they are refinement of each other;
- *Independence*. Two partitions  $\zeta$  and  $\zeta'$  are independent if  $\mu[C \cap C'] = \mu[C] \cdot \mu[C']$  whenever  $C \in \zeta$  and  $C' \in \zeta'$ ;
- If  $T : X \rightarrow X$  is a transformation, we shall write  $T^{-1}(X) = \{T^{-1}(C_i) : 1 \leq i \leq m\}$ .

We can define a metric on the space of partitions we shall denote  $\wp(X)$ . Define  $d : \wp(X) \rightarrow \mathbb{R}^+$  by  $d(\zeta, \zeta') = \min_{\sigma \in S_m} \sum_{i=1}^m \mu [C_i \Delta C'_{\sigma(i)}]$  where  $\Delta$  denotes symmetric difference, and  $\zeta = \{C_i : 1 \leq i \leq m\}$ ,  $\eta = \{D_i : 1 \leq i \leq m\}$ , and finally  $S_m$  is the group of permutation of  $\{1, \dots, m\}$ . Indeed,  $d$  is a distance. (i) Given  $\zeta = \{C_i : 1 \leq i \leq m\}$  and  $\zeta' = \{C'_i : 1 \leq i \leq m\}$ , if  $d(\zeta, \zeta') = 0$  then  $\sum_{i=1}^m \mu [C_i \Delta C'_{\sigma_0(i)}] = 0$  for some  $\sigma_0 \in S_m$ . By positivity, this must imply that  $\mu [C_i \Delta C'_{\sigma_0(i)}] = 0$  for all  $1 \leq i \leq m$ . This means that for each  $i$ , there exists  $j = \sigma_0(i)$  such that  $C_i$  is essentially subset an element in  $\zeta'$ , namely  $C'_j$ . Similarly, given  $i$ , there exists  $j = \sigma_0^{-1}(i)$  such that  $C'_i$  is essentially subset of  $C_j$ . Hence  $\zeta$  and  $\zeta'$  are equivalent. Now, let us prove the triangle inequality. Given three partitions  $\zeta$ ,  $\zeta'$  and  $\zeta''$  whose elements are denoted  $C_i, C'_i, C''_i$  respectively we have for some  $\sigma_0 \in S_m$

$$d(\zeta, \zeta') = \min_{\sigma \in S_m} \sum_{i=1}^m \mu [C_i \Delta C'_{\sigma(i)}] = \sum_{i=1}^m \mu [C_i \Delta C'_{\sigma_0(i)}]$$

Given  $\sigma, \sigma' \in S_m$ , we have  $\sum_{i=1}^m \mu [C_i \Delta C'_{\sigma(i)}] = \sum_{i=1}^m (\mu [C_i] + \mu [C'_{\sigma(i)}] - 2\mu [C_i \cap C'_{\sigma(i)}]) = 2 \left(1 - \sum_{i=1}^m \mu [C_i \cap C'_{\sigma(i)}]\right)$  and  $\sum_{i=1}^m \mu [C'_i \Delta C''_{\sigma'(i)}] = 2 \left(1 - \sum_{i=1}^m \mu [C'_i \cap C''_{\sigma'(i)}]\right)$ . Let  $i_0$  and  $\ell_0 = \sigma(i_0)$  be two distinct indices such that  $\mu [C_i \cap C'_{\sigma(i)}] \leq \mu [P_i \cap C'_{\ell_0}]$  and  $\mu [C'_i \cap C''_{\sigma'(i)}] \leq \mu [C'_{i_0} \cap Q_i]$  for all  $i$  and for some partitions  $\{P_i : 1 \leq i \leq m\}$  and  $\{Q_i : 1 \leq i \leq m\}$ . Then, with  $\sum_j \mu [C_j \Delta C''_{\tau(j)}] = 2 \left(1 - \sum_j \mu [C_j \cap C''_{\tau(j)}]\right)$ , this gives

$$\begin{aligned} \sum_i \mu [C_i \Delta C'_{\sigma(i)}] + \sum_i \mu [C'_i \Delta C''_{\sigma'(i)}] - \sum_i \mu [C_i \Delta C''_{\tau(i)}] &\geq 2 \left(1 - \sum_i \mu [P_i \cap C'_{\ell_0}] - \sum_i \mu [C'_{i_0} \cap Q_i] \right. \\ &\quad \left. + \sum_j \mu [C_j \cap C''_{\tau(j)}]\right) \\ &= 2 \left(1 - \mu [C'_{\ell_0}] - \mu [C'_{i_0}] + \sum_j \mu [C_j \cap C''_{\tau(j)}]\right) \geq 0 \end{aligned}$$

The last inequality is from the fact that  $\zeta' = \{C'_i\}_{1 \leq i \leq m}$  is a partition of  $X$ , implying that  $\mu [C'_{i_0}] + \mu [C'_{\ell_0}] \leq \sum_i \mu [C'_i] = 1$ .

Since this is true for all  $\sigma, \sigma'$  and  $\tau$ , we can take the infimums which are attained at some  $\sigma_0, \sigma'_0$  and  $\tau_0$  therefore establishing the triangle inequality.

What motivates the definition of metric entropy follows from the Bernoulli automorphism of the shift space  $\Sigma_m$ . The metric entropy for a partition  $\zeta = \{C_1, \dots, C_m\}$  is defined by

$$H(\zeta) = - \sum_{i=1}^m \mu [C_i] \log \mu [C_i]$$

If for  $x \in X$  we consider  $m(x, \zeta)$  the element  $C_x$  of  $\zeta$  containing  $x$ , then  $H(\zeta) = -\sum_{i=1}^m \int_{x \in C_i} \log \mu[C_i] d\mu = -\sum_{i=1}^m \int_{x \in C_i} \log m(x, \zeta) d\mu = -\int_X \log m(x, \zeta) d\mu$ .

**Proposition.** [2] *Let  $\xi$  and  $\eta$  be finite partitions. Then*

- (i)  $H(\xi) \geq 0$ ,  $H(\xi) = 0$  if and only if  $\xi$  is the trivial partition;
- (ii) if  $\xi \leq \eta$  then  $H(\xi) \leq H(\eta)$  and equality holds if and only if  $\xi$  and  $\eta$  are equivalent;
- (iii) If  $\xi$  has  $n$  elements, then  $H(\xi) \leq \log n$ , and equality holds if and only if each element has measure  $\frac{1}{n}$ ;
- (iv)  $H(\xi \vee \eta) \leq H(\xi) + H(\eta)$  with equality if and only if  $\xi \perp \eta$ .

*Proof.* We only prove the first statement, the rest is found in [2]. Let  $\xi = \{C_i : 1 \leq i \leq m\}$  be a partition for  $X$ . It is obvious that  $H(\xi) \geq 0$ . If  $H(\xi) = 0$  then  $\int_X -\log m(x, \xi) d\mu = 0$  with  $-\log m(x, \xi) \geq 0$ . Since  $\mu[X] = 1$ , it must be true that  $-\log m(x, \xi) = 0$  for all  $x \in X$ , then  $m(x, \xi) = 1$  that is, any element of  $\xi$  contains essentially the whole set  $X$ . In other words,  $\nu := \{X\}$  is a refinement of  $\xi$ . Therefore,  $\xi \sim \nu$ .  $\square$

## 1.2 Conditional Entropy

Conditional entropy for a finite partition  $\xi$ ,  $|\xi| = I$ , with respect to finite partition  $\eta$ ,  $|\eta| = J$  is defined as

$$H(\xi|\eta) = -\sum_{j \in J} \mu[D_j] \sum_{i \in I} \mu[C_i|D_j] \log \mu[C_i|D_j] = -\int_X \log \mu[C(x)|D(x)] d\mu[x].$$

Remark:  $\rho(\xi, \eta) = H(\xi|\eta) + H(\eta|\xi)$ , is called Rokhlin metric.

## 1.3 Entropy of a measure-preserving transformation

Let  $T : X \rightarrow X$  be a measure-preserving transformation and  $\zeta = \{C_\alpha : \alpha \in I\}$  be a partition of  $X$  with finite entropy. Consider  $T^{-k}(\zeta) = \{T^{-k}(C_\alpha) : \alpha \in I\}$  and

$$\zeta^n = \zeta \vee T^{-1}(\zeta) \vee \dots \vee T^{-n+1}(\zeta)$$

Since  $\mu[T^{-k}(C_i)] = \mu[C_i]$  (measure preserving), also since  $H(T^{-k}(\zeta)) = H(\zeta)$ , and  $H(\xi \vee \eta) \leq H(\xi) + H(\eta)$ , it follows that

$$\begin{aligned}
H(\zeta^{m+n}) &\leq H(\zeta \vee T^{-1} \vee \dots \vee T^{-m+1}(\zeta) \vee T^{-m}(\zeta) \vee \dots \vee T^{-m-n+1}(\zeta)) \\
&\leq H(\zeta^m) + H\left(\bigvee_{i=m}^{m+n-1} T^{-i}(\zeta)\right) \\
&= H(\zeta^m) + H\left(\bigvee_{i=0}^{n-1} T^{-i-m}(\zeta)\right) \\
&\leq H(\zeta^m) + \bigvee_{i=0}^{n-1} H(T^{-i-m}(\zeta)) \\
&= H(\zeta^m) + \bigvee_{i=0}^{n-1} H(T^{-i}(\zeta)) \\
&= H(\zeta^m) + H(\zeta^n)
\end{aligned}$$

By Fekete's subadditivity lemma, we know

$$h(T, \zeta) = \lim_{n \rightarrow \infty} \frac{1}{n} H(\zeta^n)$$

exists, and is called *metric entropy* (or *measure-theoretic entropy*) of  $T$  with respect to the partition  $\zeta$ . Note by subadditivity that for all  $n \in \mathbb{N}$ ,  $H(\zeta^n) \leq nH(\zeta)$ , therefore  $\frac{1}{n}H(\zeta^n) \leq H(\zeta)$  so that  $H(T, \zeta) \leq H(\zeta)$ .

**Proposition.**  $h(T, \zeta) = \lim_{n \rightarrow \infty} H(\zeta|T^{-1}(\zeta^n))$ .

The metric entropy of  $T$  is defined as the supremum of  $h(T, \zeta)$  over all finite measurable partitions  $\zeta$  of  $X$ :

$$h(T) := \sup \{h(T, \zeta) : \zeta \text{ is measurable finite partition of } X\}.$$

*Entropy is an invariant for isomorphic dynamical systems[3]: it was first introduced by Kolmogorov in 1958 before Sinai brought slight modifications. In 1970, D.S. Ornstein showed that entropy is a complete invariant for invertible Bernoulli shifts, that is, two Bernoulli shifts are measure theoretic isomorphic if and only if they have the same entropy.*

## 1.4 Kolmogorov-Sinai Theorem

**Definition.** [Refining and generating partition]

- $\{\zeta_n\}$  is refining if  $\zeta_{n+1} \geq \zeta_n$ ;
- $\{\zeta_n\}$  is generating if  $\forall \xi \forall \delta > 0 \exists n_0$  such that for all  $n \geq n_0$ , there exists  $\xi_n \leq \bigvee_{k=-n}^n \zeta_n$  with  $d(\xi_n, \xi) < \delta$ ;
- A generator is a finite partition  $\xi$  such that the sequence  $\bigvee_{k=0}^n T^k(\xi)$  is generating.

Lebesgue space has a generating sequence of finite partitions.

**Theorem** (Kolmogorov-Sinai, [2]). *Let  $\xi$  be a generator for  $T$ . Then  $h(T) = h(T, \xi)$ .*

## 2 Elements of Solution

### 2.1 Variational Principle

in this section  $f : X \rightarrow X$  is a homeomorphism,  $X$  is compact and  $\mathcal{M}$  is the Borel probability measure space on  $X$ , that is, the space of all finite Borel measures  $\mu$  with  $\mu[X] = 1$ .

**Lemma.** *Let  $\mu, \nu \in \mathcal{M}$ . For any measurable partition  $\xi$  of  $X$*

$$tH_\mu(\xi) + (1-t)H_\nu(\xi) \leq H_{t\mu+(1-t)\nu}(\xi)$$

*Proof.* For each  $n \in \mathbb{N}$ ,  $tH_\mu(\xi^n) + (1-t)H_\nu(\xi^n) = -t \sum_i \mu[C_{i,n}] \log \mu[C_{i,n}] - (1-t) \sum_i \nu[C_{i,n}] \log \nu[C_{i,n}] = -\sum_i (t\mu[C_{i,n}] \log \mu[C_{i,n}] + (1-t)\nu[C_{i,n}] \log \nu[C_{i,n}])$ . Since  $x \mapsto x \log x$  is convex, it follows that  $t\mu[C_{i,n}] \log \mu[C_{i,n}] + (1-t)\nu[C_{i,n}] \log \nu[C_{i,n}] \geq (t\mu[C_{i,n}] + (1-t)\nu[C_{i,n}]) \log (t\mu[C_{i,n}] + (1-t)\nu[C_{i,n}])$ . Then  $tH_\mu(\xi^n) + (1-t)H_\nu(\xi^n) \leq -\sum_i (t\mu[C_{i,n}] + (1-t)\nu[C_{i,n}]) \log (t\mu[C_{i,n}] + (1-t)\nu[C_{i,n}]) = H_{t\mu+(1-t)\nu}(\xi^n)$ . Therefore,  $tH_\mu(\xi) + (1-t)H_\nu(\xi) \leq H_{t\mu+(1-t)\nu}(\xi)$ .  $\square$

Given a partition  $\xi = \{A_1, \dots, A_k\}$ , define its boundary as  $\partial\xi = \bigcup_{1 \leq i \leq k} \partial A_i$  where  $\partial A = \overline{A} \setminus \text{int}(A) = \overline{A} \cap X \setminus A$  is the topological boundary in the ordinary sense.

**Lemma.** *Let  $\mu \in \mathcal{M}$ .*

1. *For any  $x \in X$  and any  $\delta > 0$ , there exists  $\delta' \in (0, \delta)$  such that  $\mu[\partial B(0, \delta')] = 0$ ;*
2. *For any  $\delta > 0$ , there is a finite measurable partition  $\xi = \{C_1, \dots, C_k\}$  with  $\text{diam}(C_i) < \delta$  for all  $i$  and  $\mu[\partial\xi] = 0$ ;*
3. *If  $\{\mu_n\} \subseteq \mathcal{M}$  is a sequence of Borel probability measure that converges to  $\mu$  in the weak\* topology<sup>1</sup>, and  $A$  is a measurable set with  $\mu[\partial A] = 0$ , then  $\mu[A] = \lim_{n \rightarrow \infty} \mu_n[A]$ .*

Let  $\mathcal{M}_f$  represent the set of all  $f$ -invariant Borel probability measures. Then:

**Theorem** (Variational principle, [2]). *Let  $f$  be a homeomorphism of a compact metric space  $X$ . Then  $h_{\text{top}}(f) = \sup \{h_\mu(f) : \mu \in \mathcal{M}_f\}$ .*

### 2.2 Entropy for expansive map

From variational principle,  $h_\lambda(f) \leq h_{\text{top}}(f)$  for all  $\lambda \in \mathcal{M}_f$ . Let  $E_n$  be an  $(n, \epsilon)$ -separated set with  $\epsilon \leq \delta_0$ , where  $\delta_0$  is the expansiveness constant of  $f$ . By expansiveness, for distinct  $x, y$  in  $X$  there exists  $i \in \mathbb{Z}$  such that  $d(f^i(x), f^i(y)) \geq \delta_0$ . Define  $\nu_n = \frac{1}{|E_n|} \sum_{x \in E_n} \delta_x$  and  $\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} f_*^i \nu_n$ . By compactness, some  $\{\mu_{n_k}\}_{k \in \mathbb{N}}$  converges. Let  $\mu := \lim_k \mu_{n_k}$ , which is an accumulation point for  $\{\mu_n\}_{n \in \mathbb{N}}$ , and which is obviously  $f$ -invariant. Let  $\xi$  be a measurable partition such that  $\text{diam}(C) < \epsilon$  for all  $C \in \xi$  and  $\partial\xi = 0$ .

Consider the sequence  $\{\bigvee_{k=-n}^n f^k(\xi)\}_{n \in \mathbb{N}}$ . For each  $x, y \in \bigvee_{k=-\infty}^{\infty} f^k(\xi)$ , we have  $d(f^k(x), f^k(y)) < \epsilon \leq \delta_0$  for all  $k \in \mathbb{Z}$ . By expansiveness, it must be true that  $x = y$ . Then the maximal diameter of

<sup>1</sup>meaning  $\int h(x) \mu_n(x) \rightarrow \int h(x) d\mu(x)$  for all  $h$  measurable.

elements of  $\{V_{-n \leq i \leq n} T^i(\xi)\}$  goes to 0. Hence,  $\{V_{k=-n}^n f^k(\xi)\}$  is generating. Then  $\xi$  is generator and by Kolmogorov-Sinai, we know  $h_\mu(f) = h_\mu(f, \xi)$ .

Now, for each  $C \in \xi^n$ ,  $\nu_n[C] = 0$  or  $\frac{1}{|E_n|}$ . Then  $H_{\nu_n}(\xi^n) = \log |E_n|$ . Then, as in a lemma in [2], we can prove that  $\overline{\lim}_{n \rightarrow \infty} \log |E_n| \leq \lim_{q \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{1}{q} H_{\mu_n}(\xi^q) = h_\mu(f, \xi)$ . Therefore,  $h_{\text{top}}(f) \leq h_\mu(f)$ .

## References

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