On Brickman's theorem

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Overview

V is $\mathbb{R}$-vector space.

Complex structure on V (cf.,ncatlab.org/nlab/show/complex+structure) : $J \in \mathrm{Hom}\,(V, V)$ s.t. $J \circ J = -Id_V$.

Example : $J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$(symplectic matrix).

$\phi : \; V \longrightarrow V$ is said to be linear if $\phi \circ J = J \circ \phi$

Hermitian form on (V, J) : real-bilinear form $h : \; V \otimes V \longrightarrow \mathbb{C}$ that is sesquilinear : (i) h is complex-linear in first argument ; (ii) $h\,(v, u) = h\,(u, v)^*$.

- **Positive definiteness :** h is positive definite if (1) $h(v, v) \geqslant 0$ and (2) $h(v, v) = 0$ if and only if $v = 0$. Then the complex vector space $((V, J), h)$ is called a **Hermitian space.**

- If *complete* : it's a Hilbert space.

- Define $g(-,-) = \mathrm{Re}(h(-,-))$ and $w(-,-) = \mathrm{Im}(h(-,-))$. Then $G: \ V \otimes V \longrightarrow \mathbb{R}$ is a real bilinear form while $\omega: \ V \wedge V \longrightarrow \mathbb{R}$ is an alternating real-bilinear form.

- *Fact* : $h = g - i\omega$
- $\mathrm{Spec}\, J = \{\pm i\}$ (J extension on $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$.)

$C \in M_n(\mathbb{C})$.

- **Initial investigation :** On the field of values of a matrix (Louis Brickman)
- $W(C) := \{\langle C\overline{x}, x\rangle : x \in \mathbb{C}^n, \|x\| = 1\}$

$W(C)$ is ***compact*** and ***connected.*** $Q: \; x \longmapsto \langle C\overline{x}, x\rangle$ continuous ;

then $Q\left(x^{(k)}\right) \longrightarrow y \in \mathbb{C}$ for some $\left\{x^{(k)}\right\} \subset W(C)$ converging to $x^{(\infty)} \in \mathbb{C}^n$ implies $y = Q\left(x^{(\infty)}\right)$. Since $\partial B(0,1)\,(= S^{n-1})$ compact : $x^{(\infty)} \in \partial B(0,1)$. Additionally, $W(C) = Q(\partial B(0,1))$ is connected.

Is $W(C)$ convex ?

- **Toeplitz** showed $W(C)$ has a "*convex outer boundary*" ;
- and later **F. Hausdorf** proved $W(C)$ itsefl is convex.
- "*Convex hull of a certain algebraic curve of degree n obtained from* C "[1]

1. cf., R. Kippenhahn, , *Über den Wertevorrat einer Matrix*, Math. Nachr. vol. 6 (1951) pp. 193-228

Let $C = A + iB$ be the *unique Hermitian decomposition*, then we obtain :

$$W(A,B) := \{\langle A\overline{x}, x\rangle, \langle B\overline{x}, x\rangle : x \in \mathbb{C}^n,\ \|x\| = 1\}$$

What about
$W(M,N,L) := \{(\langle M\overline{x}, x\rangle, \langle N\overline{x}, x\rangle, \langle L\overline{x}, x\rangle) : x \in \mathbb{C}^n,\ \|x\| = 1\}$? **No longer convex !** (Brickman, boundary)

Real analog :

$$R(A,B) = \{(\langle Ax, x\rangle, \langle Bx, x\rangle) : x \in \mathbb{R}^n,\ \|x\| = 1\}$$

where $A, B \in M_n(\mathbb{R})$ are symmetric positive definite matrices.

- Brickman ([1]) : for $n \geqslant 3$, we have $R(A,B) = W(A,B)$;
- it's **true** precisely if $R(A,B)$ is **convex.**

Brickman's original proof

$X := R(A,B) = \{(\langle Ax,x\rangle,\langle Bx,x\rangle) : x \in \mathbb{R}^n, \|x\| = 1\}.$

Theorem

If $n \geqslant 3$, X is convex.

Proof.

Idea : study the inverse image under
$\partial B(0,1) \ni\ x \longmapsto (\langle Ax,x\rangle,\langle Bx,x\rangle)$ of any line $as+bt+c=0$ and show it intersects the $st-$plane into a connected subset of $\partial B(0,1)$.

- Work within $P^{n-1}(\mathbb{R})$; identify $[x]$ with x.
- Define $\Phi :\ x \longmapsto (\xi(x), \eta(x))$ where $\xi(x) := \frac{\langle Ax,x\rangle}{\|x\|^2}$ and $\eta(x) := \frac{\langle Bx,x\rangle}{\|x\|^2}$(independent or representative).
- $\Phi\left(P^{n-1}(\mathbb{R})\right) \subset X.$

- The preimage of any line $as + bt + c = 0$ in st plane is $\left\{ x \in P^{n-1}(\mathbb{R}) : a\frac{\langle Ax,x\rangle}{\|x\|^2} + b\frac{\langle Bx,x\rangle}{\|x\|^2} + c = 0 \right\}$. We have

$$a\frac{\langle Ax,x\rangle}{\|x\|^2} + b\frac{\langle Bx,x\rangle}{\|x\|^2} + c = 0 \iff \frac{\langle aAx,x\rangle + \langle bBx,x\rangle + \langle cx,x\rangle}{\|x\|^2} = 0$$
$$\iff \langle (aA + bB + cI)\, x, x\rangle = 0$$

Need to prove the hyperconic is connected!

The connectedness is shown by Induction.

- The base case $n = 3$ corresponds to the ordinary projective conic which is connected.
- Suppose : all Q_{n-1} connected in $P^{n-1}(\mathbb{R})$, $n \geqslant 3$
- $Q_n \cap \mathcal{H}$ is a hyperconic Q_{n-1} which is connected, where $\mathcal{H}$ is any hyperplane through two arbitrary points in Q_n. □

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Remark (i) Brickman provides a counter-example dismissing the case associated to three hermitian forms(cf., op.cit., p.63) ; (ii) *Issue* : use of projective space $P^{n-1}(\mathbb{R})$.

Modern proof

- Due to Juan Enrique Martínez-Legaz ([4]) ;
- **Idea** : '*Euclideanize*' the proof of the convexity of X.

Lemma

Let $S \in M_3(\mathbb{R})$ *and* $\gamma \in \mathbb{R}$. *Then*

$$Z := \{x \in \mathbb{R}^n : \|x\| = 1,\ \langle Sx, x\rangle = \gamma\} = Y \cup (-Y)$$

where Y is connected and $-Y := \{-y : y \in Y\}$ is the antipodal copy of Y.

- Let $Q: \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $Q(x) = \langle Sx, x \rangle$. Q is continuous. We have $Z = \partial B(0,1) \cap Q^{-1}(\gamma)$;
- The level set $Q^{-1}(\gamma)$ is a quadric surface;
- Wlog S is symmetric otherwise use $A = \frac{S+S^T}{2}$; $S = \text{diag}(\lambda_1, \lambda_2, \lambda_3)$ where λ_i eigenvalues.

Fact : $Q^{-1}(\gamma)$ has at most two connected components. Setting $t_i = x_i^2$, $1 \leqslant i \leqslant 3$, yields the 2-simplex $\left\{ t = (t_1, t_2, t_3) \geqslant 0 : \sum_{i=1}^{3} t_i = 1 \right\}$.

The solution set of $\sum_{k=1}^{3} \lambda_k t_k = \gamma$ (linear equation) in this simplex is therefore either $\emptyset$, a single point, a line segment or the whole 2-simplex. 'Pulling back' through $t_i \mapsto \pm\sqrt{t_i}$ yields at most two connected components.

Consequence : Z has at most two connected components. In the case where Z is disconnected, Z must be symmetric under $x \longmapsto -x$ therefore has two antipodal connected components. $\square$

Back to Brickman. Let V be an i.p.s., with $\dim(V) \geqslant 3$, and let $A, B \in \operatorname{Hom}(V, V)$ be given. Then

$$C = \{(\langle Ax, x\rangle, \langle Bx, x\rangle) : x \in V,\ \|x\| = 1\}$$

is convex.

Proof. Case $n = 3$: wlog $V = \mathbb{R}^3$. Consider any arbitrary line $L : \alpha s + \beta t + \gamma = 0$ in the st plane.

- What's $C \cap L$?
- $(\langle Ax, x\rangle, \langle Bx, x\rangle)$ belongs to L iff $0 = \alpha\langle Ax, x\rangle + \beta\langle Bx, x\rangle + \gamma = \langle(\alpha A + \beta B + \gamma I)x, x\rangle$. Setting $S_{\alpha,\beta,\gamma} := \alpha A + \beta B + \gamma I$, we have that $C \cap L = \{(\langle Ax, x\rangle, \langle Bx, x\rangle) : \|x\| = 1, \langle S_{\alpha,\beta,\gamma}x, x\rangle = 0\} =: X_{\alpha,\beta,\gamma}$.

- $T : \mathbb{R}^3 \supset S^2 \to \mathbb{R}^2$ defined by $T(x) = (\langle Ax, x \rangle, \langle Bx, x \rangle)$ (continuous)
- $X_{\alpha,\beta,\gamma} = Y_{\alpha,\beta,\gamma} \cup (-Y_{\alpha,\beta,\gamma})$
- We have that $C \cap L = T(X_{\alpha,\beta,\gamma}) = T(Y_{\alpha,\beta,\gamma})$.

General case. $\dim V \geqslant 3$. Let $\big(\langle Ax^i, x^i \rangle, \langle Bx^i, x^i \rangle\big)$, $i = 1, 2$ be any two points of C and let $\lambda \in [0, 1]$ be given.

- Let $\{w^1, w^2\}$ be an ONB spanned by $x^1, x^2 \in \text{span}\{w^1, w^2\}$ or any ONB containing x_1 when x^1 and x^2 are colinear. Pick $w^3 \in (\text{span}\{w^1, w^2\})^\perp$ with $\|w^3\| = 1$.
- Consider $W := \text{span}\{w^1, w^2, w^3\}$, $\widetilde{A} = (a_{ij})$, $\widetilde{B} = (b_{ij}) \in M_3(\mathbb{R})$ where $a_{ij} = \langle Aw^i, w^j \rangle$ and $b_{ij} = \langle Bw^i, w^j \rangle$.
- Define $\varphi : \; W \longrightarrow \mathbb{R}^3$ by $\varphi(w^i) := e^i$(an isometry : ONB onto an ONB.)
- On the one hand we have $\langle Aw^i, w^j \rangle = a_{ij}$. On the other, we have $\langle \widetilde{A}\varphi(w^i), w^j \rangle = \langle \widetilde{A}e_i, e_j \rangle = a_{ji} = a_{ij}$(symmetric ?).
- Thus $\langle Aw, w \rangle = \langle \widetilde{A}\varphi(w), w \rangle$ and $\langle Bw, w \rangle = \langle \widetilde{B}\varphi(w), w \rangle$

- $\left\{\left(\langle\widetilde{A}u,u\rangle,\langle\widetilde{B}u,u\rangle\right) : u\in\mathbb{R}^3,\ \|u\|=1\right\}$ is convex ;
- then

$$
\begin{aligned}
(1-\lambda)&\left(\langle\widetilde{A}\varphi\left(x^1\right),\varphi\left(x^1\right)\rangle,\langle\widetilde{B}\varphi\left(x^1\right),\varphi\left(x^1\right)\rangle\right)\\
&\quad+\lambda\left(\langle\widetilde{A}\varphi\left(x^2\right),\varphi\left(x^2\right)\rangle,\langle\widetilde{B}\varphi\left(x^2\right),\varphi\left(x^2\right)\rangle\right)\\
&=\left(\langle\widetilde{B}\left(u\right),u\rangle,\langle\widetilde{B}u,u\rangle\right)\text{ for some } u\in\mathbb{R}^3\text{with } \|u\|=1.
\end{aligned}
$$

- Use $\langle *x^i,x^i\rangle=\langle\widetilde{*}\varphi\left(x^i\right),\varphi\left(x^i\right)\rangle$ and $\langle\widetilde{*}u,u\rangle=\langle *\varphi^{-1}\left(u\right),\varphi^{-1}\left(u\right)\rangle$ to conclude, where $*=A$ or B.

Concluding remarks

Paper : Permanently Going Back and Forth between the "Quadratic World" and the "Convexity World" in Optimization[3]&[2]

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