Ethiopian Multiplication

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Main Text

Let n and m be two given integers. We want to compute $n \times m$ using the Ethiopian merchants method.

In a youtube channel[1], it is shown that to multiply n by m, one proceeds as follows.

n	n_1	n_2	$\cdots$	n_ℓ	$\cdots$
m	m_2	m_3	$\cdots$	m_ℓ	$\cdots$

where $\ell \in \mathbb{N}$, $n_\ell = \left\lfloor \frac{n_{\ell-1}}{2} \right\rfloor$, $m_\ell = 2m_{\ell-1}$. We stop the algorithm whenever we reach $n_\ell = 1$ for some ℓ. The following example illustrates the process. Take $n = 72$ and $m = 14$.

72	36	18	9	4	2	1
14	28	56	112	224	448	896

Looking at the first row, eliminate any column which has an even entry on this first row. Thus, we eliminate the first, the second, the fourth and the fifth columns. Now add the numbers on the second row which correspond to the remaining columns, namely $112 + 896 = 1008$. If instead we multiply 14 by 72, we will get the following.

14	7	3	1
72	144	288	576

This time, no column is to be eliminated. Thus, $14 \times 72 = 144 + 288 + 576 = 1008$ which is exactly what we've got earlier.

We now give the formal proof which justifies this algorithm. Let $r_\ell = n_{\ell-1} - 2n_\ell$ for all $\ell \in \mathbb{N}$ with the convention that $n_0 = n$. Then $r_\ell \in \{0, 1\}$. Moreover we have that $n = 2n_1 + r_1 = 4n_2 + 2r_2 + r_1$. By induction, we get

$$n = 2^\ell n_\ell + \sum_{k=0}^{\ell-1} 2^k r_{k+1}. \tag{$*$}$$

When for some $\ell = j \in \mathbb{N}$ we have $n_j = 1$, the expression $(*)$ becomes

[1] @AllThisMath

$$n = 2^j + \sum_{k=0}^{j-1} 2^k r_{k+1} \tag{$**$}$$

Using $(**)$, we have

$$n \times m = \left(2^j + \sum_{k=0}^{j-1} 2^k r_{k+1}\right) \times m = 2^j m + \sum_{k=0}^{j-1} 2^k r_{k+1} m$$

But we know that $2^\ell m = m_\ell$ for all $\ell \in \mathbb{N}$. In particular, for each of $k \in \{0, 1, \ldots, j-1\}$ we have $2^k m = m_k$ with the convention that $m_0 = m$. Then

$$n \times m = m_j + \sum_{k=0}^{j-1} r_{k+1} m_k.$$

This shows that whenever the coefficient before m_k, $0 \leqslant k \leqslant j-1$ is zero, that is whenever $r_{k+1} = 0$ which is equivalent to whenever n_k is **even**[2] the corresponding term contributes nothing to the multiplication, thus confirming the deeds of the Ethiopian merchants. If for example $r_1 = \cdots = r_j = 0$, the product is simply m_j. This is the case when $n = 2^j$. Finally, we get similar result when we interchange n and m which must provide the same product since multiplication is commutative.

[2] $n_k = 2n_{k+1} + r_{k+1} = 2n_{k+1}$ if and only if $r_{k+1} = 0$