Analysis, Clemson Preliminary Exam 2022

Hamilijaona Elmar

Problem 1

Let $\mathcal{X}$ be a normed linear space(n.l.s)Let $\{x_n\}_{n\geqslant 1}$ such that $x_n \to x$ and $T_n : \mathcal{X} \to \mathcal{X}$ is a sequence of bounded linear operators such that $T_n \to T$ in operator norm. Prove that $T_n x_n \to Tx$.

Solution. For each $n \in \mathbb{N}$, we have by triangle inequality:

$$\|T_x x_n - Tx\| \leqslant \|T_n x_n - T_n x\| + \|T_n x - Tx\| \leqslant \|T_n\| \|x_n - x\| + \|T_n - T\| \|x\|$$

Since $T_n \to T$ in operator norm, the sequence $\{T_n\}$ is bounded. There exists $M > 0$ such that $\|T_n\| \leqslant M$. Then $\|T_n x_n - Tx\| \leqslant M \|x_n - x\| + \|T_n - T\| \|x\|$ with $\|x_n - x\| \to 0$ and $\|T_n - T\| \to 0$. Hence $\|T_n x_n - Tx\| \to 0$ when $n \to \infty$.

Problem 2

Let $T : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. Denote by $\operatorname{Ran}(T)$ the range of T.Suppose that there exists $c > 0$ such that $c\|x\| \leqslant \|Tx\|$ for all $x \in \mathcal{H}$. Prove that $\operatorname{Ran}(T)$ is a closed subspace of $\mathcal{H}$.

Solution. We know already that $\operatorname{Ran}(T)$ is a subspace of $\mathcal{H}$ since $\operatorname{Ran}(T) = T(\mathcal{H})$ with T being a linear operator. Notice that T is $1-1$ linear map: if $Tx = 0$ for some $x \in \mathcal{H}$, then $c\|x\| \leqslant \|Tx\| = 0$ implies $x = 0$. Therefore, the bounded linear operator $\widetilde{T} : \mathcal{H} \to \operatorname{Ran}(T)$ defined by $\widetilde{T}(x) = T(x)$ is bijective. Its inverse $S : \operatorname{Ran}(T) \to \mathcal{H}$ is also a bounded linear operator as shown in the following. For all $x \in \mathcal{H}$, $c\|S(Tx)\| \leqslant \|TS(Tx)\| = \left\|\widetilde{T}S(Tx)\right\| = \|Tx\|$. Hence $\|S\| \leqslant c^{-1} < \infty$. Hence $\widetilde{T}$ is a homeomorphism. Consequently, $\operatorname{Ran}(T) = \widetilde{T}(\mathcal{H})$ is closed.

Remark: $\mathcal{H}$ is automatically closed since it is the parent space and moreover a Hilbert space which must be complete, hence closed.

Problem 3

Let $C^1([-1,1])$ be the set of all continuously differentiable real-valued functions on $[-1,1]$. Consider the following two norms on $C^1([-1,1])$

$$\|f\|_\infty = \sup_{x\in[-1,1]} |f(x)| \quad \|f\|_1 = \|f\|_\infty + \|f'\|_\infty .$$

Define a linear function $T : C^1([-1,1]) \to \mathbb{R}$ by $T(f) = f'(0)$.

(a) Prove that T is not bounded if $C^1([-1,1])$ is equipped with the supremum norm $\|f\|_1$.
(b) Prove that T is bounded if $C^1([-1,1])$ is equipped with the norm $\|f\|_1$ defined above and compute its operator norm.

Solution. (a) Consider the sequence of function $\{f_n\}_{n=1}^{\infty}$ defined for each $n \in \mathbb{N}$ by $f_n(t) = \sin(\pi n t)$ for all $t \in [-1,1]$. Plainly $f_n \in C^1([-1,1])$ for each $n \in \mathbb{N}$. If there existed a $\alpha > 0$ such that $|Tf_n| \leqslant \|f_n\|_\infty$ then it would follow that $\pi n \leqslant \alpha$ for each $n \in \mathbb{N}$. This manifestly cannot happen. Hence T cannot be bounded on $(C^1([-1,1]), \|\cdot\|_\infty)$.

(b) Now consider $(C^1([-1,1]), \|\cdot\|_1)$. For all $f \in C^1([-1,1])$, we have $|Tf| = |f'(0)| \leqslant \sup_{x\in[-1,1]} |f'(x)| \leqslant \|f'\|_\infty + \|f\|_\infty = \|f\|_1$. Thus $\|T\| \leqslant 1$. Additionally, the sequence of functions $g_n : t \mapsto \sin(nt)$ satisfies for each $n \in \mathbb{N}$,

$$|Tg_n| = |g_n'(0)| = n\,, \quad \text{and } \|g_n\|_1 = \|g_n\|_\infty + \|g_n'\|_\infty = 1 + n.$$

Then for each $n \in \mathbb{N}$, $\|T\| = \sup_{f\neq 0} \frac{|Tf|}{\|f\|_1} \geqslant \frac{|Tg_n|}{\|g_n\|_1} = \frac{n}{1+n}$. Letting $n \to \infty$, we have that $\|T\| \geqslant 1$.

Problem 4

Let $\mathcal{H}$ be a Hilbert space. Let $U : \mathcal{H} \to \mathcal{H}$ be a unitary operator, i.e., bounded linear operator satisfying $UU^* = U^*U = I$. Let $\mathcal{K} = \{x \in \mathcal{H} : Ux = x\}$ be the subspace of invariant vectors of U.

(1) Prove that $\mathcal{K} = (\mathrm{Ran}\,(I-U))^\perp$.
(2) Prove that $\mathcal{H} = \mathcal{K} \oplus \overline{\mathrm{Ran}\,(I-U)}$.
(3) Let $P : \mathcal{K} \to \mathcal{K}$ be the orthogonal projection onto $\mathcal{K}$.Prove that for each $x \in H$ we have

$$Px = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} U^n x.$$

Solution. (1) Consider the map $T : \mathcal{H} \to \mathcal{H}$ by $T(x) = x - Ux = (I-U)(x)$. T is a bounded linear operator on $\mathcal{H}$. We have $\mathcal{K} = \ker(T) = \ker(I-U)$, and since U is a unitary operator, $\ker(I-U) = \ker(I-U^*) = \ker(T^*) = (\mathrm{Ran}\,(T))^\perp$. Hence $\mathcal{K} = (\mathrm{Ran}\,(T))^\perp = (\mathrm{Ran}\,(I-U))^\perp$.

(2) Since $T = I - U$ is a bounded operator, we have $\mathcal{H} = \ker(T) \oplus (\ker(T))^\perp$. But $\ker(\mathrm{T})^\perp = \ker(T^*)^\perp = \overline{\mathrm{Ran}\,((T^*)^*)} = \overline{\mathrm{Ran}\,(T)} = \overline{\mathrm{Ran}\,(I-U)}$.

(3) Let $x \in H$ be given. There exist $y \in K$ and $z \in \overline{\mathrm{Ran}\,(I-U)}$ such that $x = y + z$. We have $Px = Py + Pz = y + Pz$ with $Pz \neq 0$. Let $\epsilon > 0$ be arbitrary. There exists $z_0 = \xi - U\xi \in \mathrm{Ran}\,(I-U)$ such that $\|z - z_0\| < \epsilon$. For each $N \in \mathbb{N}$,

$$\frac{1}{N}\sum_{n=1}^{N} U^n z_0 = \frac{1}{N}\sum_{n=1}^{N} \left\{U^n\xi - U^{n+1}\xi\right\} = \frac{1}{N}\left\{U\xi - U^{N+1}\xi\right\}.$$

Then $\left\|\frac{1}{N}\sum_{n=1}^{N}U^n z_0\right\| \leqslant \frac{1}{N}\left\|U\xi - U^{N+1}\xi\right\| \leqslant \frac{2\|\xi\|}{N}$.[Because U unitary implies for each $i \in \mathbb{N}$ $\|U^i x\|^2 = \langle U^i x, U^i x\rangle = \langle U^{i-1}x, U^* U^i x\rangle = \langle U^{i-1}x, U^{i-1}x\rangle = \|U^{i-1}x\|$ which implies by induction $\|U^i x\| = \|x\|$.] Thus $\left\|\frac{1}{N}\sum_{n=1}^{N}U^n z_0\right\| < \epsilon$ whenever $N \geqslant n_0$ for some positive integer n_0. Then for all $N \geqslant n_0$, and since $U^i(Px) = Px$ for each $i \in \mathbb{N}$:

$$\begin{aligned}\left\|\frac{1}{N}\sum_{n=1}^{N}U^n x - Px\right\| &= \left\|\frac{1}{N}\sum_{n=1}^{N}U^n(x - Px)\right\| = \left\|\frac{1}{N}\sum_{n=1}^{N}U^n z\right\| \\ &\leqslant \left\|\frac{1}{N}\sum_{n=1}^{N}U^n(z - z_0)\right\| + \left\|\frac{1}{N}\sum_{n=1}^{N}U^n z_0\right\| < \left\|\frac{1}{N}\sum_{n=1}^{N}U^n\right\| \|z - z_0\| + \epsilon < \epsilon + \epsilon = 2\epsilon.\end{aligned}$$

Notice that $\left\|\frac{1}{N}\sum_{n=1}^{N}U^n\right\| \leqslant \frac{1}{N}\sum_{n=1}^{N}\|U^n\| = 1$.

Problem 5

Let $(X, \mathcal{M}, \mu)$ be a finite measure space. Suppose $\{A_n\}_{n=1}^{\infty}$ is a sequence of measurable sets such that $A_{n+1} \subset A_n$ for all $n \in \mathbb{N}$.
(a) Prove that

$$\mu\left(\cap_{n=1}^{\infty}A_n\right) = \lim_{n\to\infty}\mu(A_n).$$

(b) Let be ν another finite measure on $(X, \mathcal{M})$ such that $\nu(E) = 0$ whenever $E \in \mathcal{M}$ with $\mu(E) = 0$. Prove that for each $\epsilon > 0$ there exists $\delta > 0$ such that $\mu(E) < \delta$ implies $\nu(E) < \epsilon$.

Solution. (a) The monotonicity of μ implies that for all $n \in \mathbb{N}$, $\mu(A_{n+1}) \leqslant \mu(A_n)$. Then $\{\mu(A_n)\}_{n\geqslant 1}$ is a non increasing sequence of non negative numbers, it is convergent. Since for all $n \in \mathbb{N}$, $A_n \supset \cap_{n\in\mathbb{N}}A_n$, we have $\lim_n \mu(A_n) \geqslant \mu(\cap_{n\geqslant 1}A_n)$. Moreover, consider the sequence $\{B_n\}$ defined for each $n \in \mathbb{N}$ by $B_n := X \setminus A_n$. This sequence is monotonically increasing, therefore $\sup_{n\in\mathbb{N}}\mu(B_n) = \mu(\cup_{n\geqslant 1}B_n)$. Given $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ so that $\mu(\cup_{n\geqslant 1}B_n) - \epsilon < \mu(B_{n_0})$. Since $\mu(X) < \infty$, the latter is equivalent to $\mu(X) - \mu(\cap_{n\geqslant 1}A_n) - \epsilon < \mu(X) - \mu(A_{n_0})$. Then $\mu(A_{n_0}) < \mu(\cap_{n\geqslant 1}A_n) + \epsilon$. Then for all $n \geqslant n_0$, we have $\epsilon + \mu(\cap_{n\geqslant 1}A_n) > \mu(A_{n_0}) \geqslant \mu(A_n)$. Hence $\epsilon + \mu(\cap_{n\geqslant 1}A_n) > \lim_{n\to\infty}\mu(A_n)$. Letting ϵ goes to 0, we have that $m(\cap_{n\geqslant 1}A_n) \geqslant \lim_{n\to\infty}\mu(A_n)$.

(b) By contradiction, assume there exists $\epsilon_0 > 0$ such that for all $\eta > 0$ we have $\mu(E_\eta) < \eta$ but $\nu(E_\eta) \geqslant \epsilon_0$ $(E_\eta \in \mathcal{M}.)$ In particular, for each positive integer $n \in \mathbb{N}$ we have $\mu(E_n) < \frac{1}{n}$ but $\nu(E_n) \geqslant \epsilon_0$, $(E_n \in \mathcal{M})$. Define $E = \cap_{n=1}^{\infty}E_n \in \mathcal{M}$. Consider for each $n \in \mathbb{N}$, $A_n = \cap_{i=1}^{n}E_n$. The sequence $\{A_n\}_{n\geqslant 1}$ is monotonically decreasing. Previous part allows us to have

$$\mu(E) = \mu\left(\bigcap_{n\geqslant 1}E_n\right) = \mu\left(\bigcap_{n\geqslant 1}A_n\right) = \lim_{n\to\infty}\mu(A_n) \leqslant \lim_{n\to\infty}\mu(E_n) \leqslant \lim_{n\to\infty}\left(\frac{1}{n}\right) = 0.$$

Additionally, for each $n \in \mathbb{N}$ there exists $1 \leqslant i_n \leqslant n$ such that $A_n = \cap_{1\leqslant i\leqslant n}E_n = E_{i_n}$. Then for each $n \in \mathbb{N}$, $\nu(A_n) = \nu(E_{i_n}) \geqslant \epsilon_0 > 0$. Taking the limit when n goes to ∞, we have $\lim_{n\to\infty}\nu(A_n) \geqslant \epsilon_0$. In other words, $\nu(E) \geqslant \epsilon_0$. This contradicts our hypothesis which claims that $\nu(E) = 0$ whenever $E \in \mathcal{M}$ with $\mu(E) = 0$.

Problem 6

Let $(X, \mathcal{M}, \mu)$ be a measure space. Recall that a set $E \subset X$ is said to be $\sigma-$finite if there exists a sequence $\{E_n\}_{n\geqslant 1}$ of measurable sets with $\mu(E_n) < \infty$ for all $n \in \mathbb{N}$ such that $E = \bigcup_{n=1}^{\infty} E_n$. Prove that if $f \in L^p(X, \mu)$ for $1 \leqslant p < \infty$, then the set $E := \{x \in X : f(x) \neq 0\}$ is $\sigma-$finite.

Solution. Plainly $E = \bigcup_{n=1}^{\infty} \left\{x \in X : |f(x)| \geqslant \frac{1}{n}\right\}$. Set for each $n \in$, $E_n := \left\{x \in X : |f(x)| \geqslant \frac{1}{n}\right\}$. We claim that $\mu(E_n) < \infty$ for each $n \in \mathbb{N}$. But since f is integrable, $\infty > \int_X |f|^p \, d\mu = \int_{\left\{|f|\geqslant\frac{1}{n}\right\}} |f|^p \, d\mu + \int_{\left\{|f|<\frac{1}{n}\right\}} |f|^p \, d\mu \geqslant \int_{\left\{|f|\geqslant\frac{1}{n}\right\}} |f|^p \, d\mu \geqslant n^{-p}\mu(E_n)$. Then $\mu(E_n) \leqslant n^p \int_X |f|^p \, d\mu < \infty$.

Problem 7

Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\{f_n\}_{n\geqslant 1}$ be a sequence of integrable functions and $f : X \to \mathbb{R}$ be a measurable function.

(a) Prove that if for some $\delta > 0$ we have $\int_X |f_n(x) - f(x)| \, d\mu(x) \leqslant \frac{1}{n^{1+\delta}}$ for all $n \in \mathbb{N}$, then $f_n \to f$ pointwise a.e..
(b) Prove that $\int_X |f_n(x) - f(x)| \, d\mu(x) \leqslant \frac{1}{n}$ for all $n \in \mathbb{N}$ does not in general imply $f_n \to f$ pointwise a.e..

Solution. Assume there exists $\delta > 0$ such that

$$\int_X |f_n(x) - f(x)| \, d\mu(x) \leqslant \frac{1}{n^{1+\delta}}$$

Summing over $n \in \mathbb{N}$, we have

$$\sum_{n=1}^{\infty} \int_X |f_n(x) - f(x)| \, d\mu(x) \leqslant \sum_{n=1}^{\infty} \frac{1}{n^{1+\delta}} < \infty.$$

Put differently, this becomes

$$\lim_{n\to\infty} \sum_{i=1}^{n} \int_X |f_i(x) - f(x)| \, d\mu(x) < \sum_{n=1}^{\infty} \frac{1}{n^{\delta+1}} < \infty.$$

By linearity, we have for each $n \in \mathbb{N}$, $\sum_{i=1}^{n} \int_X |f_i(x) - f(x)| \, d\mu(x) = \int_X \sum_{i=1}^{n} |f_i(x) - f(x)| \, d\mu(x)$. Additionally, the sequence $\{\sum_{i=1}^{n} |f_i(x) - f(x)|\}_{n\geqslant 1}$ is a monotonically increasing sequence of measurable functions. By Beppo Levi, we have that

$$\lim_{n\to\infty} \int_X \sum_{i=1}^{n} |f_n(x) - f(x)| \, d\mu(x) = \int_X \lim_{n\to\infty} \sum_{i=1}^{n} |f_n(x) - f(x)| \, d\mu(x) = \int_X \sum_{n=1}^{\infty} |f_n(x) - f(x)| \, d\mu(x).$$

Thus $\int_X \sum_{n=1}^{\infty} |f_n(x) - f(x)| \, d\mu(x) < \infty$, and therefore $\sum_{n=1}^{\infty} |f_n(x) - f(x)| = 0$ for a.e. x on X.Hence, $\lim_{n\to\infty} |f_n(x) - f(x)| = 0$ for a.e. x on X. In other words, $f_n \to f$ pointwise a.e. on X.

(b) $X=[0,1]$, $\mathcal{M}=\mathcal{B}([0,1])$ and $\mu=$Lebesgue measure. For each n, define $f_n := \mathbb{1}_{J_n}$ where $J_n=\left[\frac{j}{2^{k+1}},\frac{j+1}{2^{k+1}}\right]$ with $k\in\mathbb{N}$ is the unique non negative integer that satisfies $2^k\leqslant n<2^{k+1}$ and $j:=n-2^k$. We have for each $n\in\mathbb{N}$,

$$\int_X |f_n(x)|\,d\mu(x)=\int_{[0,1]}\mathbb{1}_{J_n}(x)\,d\mu(x)=\mu(J_n)=\frac{1}{2^{k+1}}<\frac{1}{n}.$$

Now let $x\in[0,1]$ be given. For all $k\in\mathbb{N}$, $[0,1]=\bigcup_{\ell=0}^{2^{k+1}-1}\left[\frac{\ell}{2^{k+1}},\frac{\ell+1}{2^{k+1}}\right]=\bigcup_{\ell=0}^{2^{k+1}-1}J_{2^k+\ell}$. Then $x\in J_{2^k+\ell_{x,k}}$ for some $0\leqslant \ell_{x,k}<2^{k+1}$. On the set $B:=\bigcup_{k\geqslant 1}J_{2^k+\ell_{x,k}}$ we have $f_{2^i+\ell_{x,i}}(x)=1\not\to 0$ as $i\to\infty$. In addition,

$$\mu(B)\geqslant \mu\left(J_{2+\ell_{x,1}}\right)=\mu\left(\left[\frac{\ell_{x,1}}{2^2},\frac{\ell_{x,1}+1}{2^2}\right]\right)=\frac{1}{4}>0.$$

Problem 8

Let $f:[1,\infty)\to\mathbb{R}$ is a continuous function such that $\lim_{x\to\infty}|f(x)|=0$. Prove that for any integrable function $g:[1,\infty)$ the following equality holds

$$\lim_{n\to\infty}\int_1^\infty f(n+x)\,g(x)=0.$$

Solution. Since $\lim_{x\to\infty}|f(x)|=0$, there exists $M>0$ such that $|f(x)|<1$ whenever $x>M$. Then for all $x\geqslant 1$, $|f(x)|\leqslant \max\left\{\sup_{t\in[1,M]}|f(t)|,1\right\}<\infty$. In other words, f is bounded. Consider the sequence $\{f_n\}_{n\geqslant 1}$ of real valued, measurable functions defined for each $n\in\mathbb{N}$ by $f_n(x)=f(n+x)\,g(x)$ for all $x\in[1,\infty)$. This sequence satisfies

$$\int_{[1,\infty)}|f_n(x)|\,d\mu(x)\leqslant M_f\int_{[1,\infty)}|g(x)|\,d\mu(x)<\infty$$

where M_f is an upper bound for $|f|$.

Moreover, for almost all $x\in[1,\infty)$, $|g(x)|<\infty$ because g is integrable. Then $f_n(x)\to 0$ poitwise a.e. on $[1,\infty)$. We can therefore apply the Lebesgue dominated convergence theorem to obtain

$$\lim_{n\to\infty}\int_{[1,\infty)}|f_n(x)|\,d\mu(x)=\int_X\lim_{n\to\infty}|f_n(x)|\,g(x)\,d\mu(x)=\int_{[1,\infty)}(0)\,d\mu(x)=0.$$

Hence we have $\lim\limits_{n\to\infty}\int_1^\infty f(n+x)\,g(x)=0$ as desired.