# Analysis, Clemson Preliminary Exam 2023

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## Problem 1

Prove the following statements.

$(a)$ If $\{x_n\}_{n\geqslant 1} \subseteq \mathbb{R}$ is a Cauchy sequence that has a convergent subsequence, then $\{x_n\}_{n\geqslant 1}$ is convergent.

$(b)$ A subset $A \subseteq \mathbb{R}$ is bounded if and only if $\lim\limits_{n} a_n x_n = 0$ for all $\{x_n\}_{n\geqslant 1} \subseteq A$ and $\{a_n\}_{n\geqslant 1} \subseteq \mathbb{R}$ with $\lim\limits_{n} a_n = 0$.

*Solution.* $(a)$ Let $\epsilon > 0$ be given. Let $\{x_{n_i}\}_{i\geqslant 1}$ be the convergent subsequence of $\{x_n\}_{n\geqslant 1}$ and let $x$ be its limit. There exists $N_1 \in \mathbb{N}$ such that $|x_{n_i} - x_n| < \frac{\epsilon}{2}$ whenever $i \geqslant N_1$. Also there exists $N_2 \in \mathbb{N}$ such that $|x_n - x_m| < \frac{\epsilon}{2}$ for all $n, m \geqslant N_2$ because $\{x_n\}_{n\geqslant 1}$ is a Cauchy sequence. Then we have for all $i \geqslant \max\{N_1, N_2\}$, $|x_i - x| \leqslant |x_i - x_{n_i}| + |x_{n_i} - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$(Remember that $n_i > i$ for all $i \in \mathbb{N}$.) Thus $\{x_n\}_{n\geqslant 1}$ converges to the same limit $x$.

$(b)$ If $A$ is bounded, there exists $C > 0$ which depends only on $A$ such that $|x| \leqslant C$ for all $x \in A$. If we take any sequence $\{x_n\}_{n\geqslant 1} \subseteq A$ and any sequence $\{a_n\}_{n\geqslant \mathbb{N}} \subseteq \mathbb{R}$ with $\lim\limits_{n} a_n = 0$, we shall have $\lim\limits_{n} |a_n x_n| \leqslant \lim\limits_{n} C\,|a_n| = 0$. Thus $\lim\limits_{n} a_n x_n = 0$. Conversely, assume the conditions are met for any such sequences. If $A$ was not bounded, everytime we choose an element $c > 0$, there would exist $x_c \in A$ such that $|x_c| > c_x$. Thus, to each $n \geqslant 1$ we could associate some $x_n \in A$ satisfying $|x_n| > n$. Define $a_n := \frac{1}{\sqrt{n}} > 0$ for $n \in \mathbb{N}$. The sequence $\{a_n\}_{n\geqslant 1} \subseteq \mathbb{R}$ satisfies $\lim\limits_{n} a_n = 0$. However, $|x_n a_n| > \frac{n}{\sqrt{n}} = \sqrt{n}$ with $\lim\limits_{n} \sqrt{n} = \infty$ making $\lim\limits_{n} |a_n x_n| = \infty$. Thus $\lim\limits_{n} a_n x_n \neq 0$ which contradicts the hypothesis. Therefore, $A$ must be bounded.

## Problem 2

Consider the Banach space $C([0,1])$ consisting of all continuous functions $f : [0,1] \to \mathbb{R}$, equipped with the supremum norm $\|f\|_\infty = \sup\limits_{x\in[0,1]} |f(x)|$. Let $K : [0,1] \times [0,1] \to \mathbb{R}$ be a continuous function. Consider the operator $T : C([0,1]) \to C([0,1])$ given by

$$Tf(x) = \int_0^1 K(x,y)\, f(y)\, dy.$$

$(a)$ Prove that $T$ is bounded.

$(b)$ Find $\|T\|$. Justify your answer.

*Solution.* $(a)$ For all $f \in C([0,1])$, we have that

$$\|Tf\|_\infty = \sup_{x\in[0,1]} |Tf(x)| \leqslant \sup_{x\in[0,1]} \left| \int_0^1 K(x,y) f(y)\, dy \right| \leqslant \left( \sup_{x\in[0,1]} \int_0^1 |K(x,y)|\, dy \right) \|f\|_\infty$$

Thus $\|T\| \leqslant \sup_{x\in[0,1]} \int_0^1 |K(x,y)|\, dy < \sup_{(x,y)\in[0,1]^2} |K(x,y)| < \infty$, making $T$ bounded.

$(b)$ From part $(a)$, we know already that $\|T\| \leqslant \sup_{x\in[0,1]} \int_0^1 |K(x,y)|\, dy$. Additionally, consider for each $x \in [0,1]$ the map $f^x$ defined by $f^x(y) = \operatorname{sgn}(K(x,y))$. Since the map sgn can be approximated by continuous functions and since $y \mapsto K(x,y)$ is continuous, the map $f^x$ actually can also be approximated by continuous functions. For example, the function $f_n^x(y) := \frac{2}{1+e^{-nK(x,y)}} - 1$, approximates $f^x$, with $\|f_n^x\|_\infty \leqslant 1$. So, yes, **the sign function of a continuous function can be approximated uniformly by a sequence of continuous functions.** Working with $\{f_n^x\}_{n\geqslant 1}$ instead of $f^x$, we get

$$|Tf_n^x(x)| = \left| \int_0^1 K(x,y) f_n^x(y)\, dy \right| \implies \lim_n |Tf_n^x(x)| = \int_0^1 K(x,y) \operatorname{sgn}(K(x,y))\, dy = \int_0^1 |K(x,y)|\, dy.$$

Taking the supremum over $x \in [0,1]$, we have that

$$\sup_{x\in[0,1]} \int_0^1 |K(x,y)|\, dy = \sup_{x\in[0,1]} \lim_n |Tf_n^x(x)| = \lim_n \sup_{x\in[0,1]} |Tf_n^x(x)| = \lim_n \sup_{x\in[0,1]} \|Tf_n^x\|_\infty \leqslant \|T\|.$$

The bound is therefore tight. Consequently, $\|T\| = \sup_{x\in[0,1]} \int_0^1 \|K(x,y)\|\, dy$.

# Problem 3

Let $\mathcal{H}$ be a Hilbert space. Recall that a sequence $\{f_n\}_{n\geqslant 1}$ converges weakly to $f \in \mathcal{H}$ if $\lim_n \langle f_n, g\rangle = \langle f, g\rangle$ for all $g \in \mathcal{H}$. We want to prove the following.

$(a)$ A sequence $\{f_n\}_{n\geqslant 1} \subseteq \mathcal{H}$ converges to $f \in \mathcal{H}$ if and only if $\lim_n \|f_n\| = \|f\|$ and $f_n$ converges weakly to $f$.

$(b)$ Let $T$ be a bounded linear operator on $\mathcal{H}$. If $\{f_n\}_{n\geqslant 1}$ converges weakly to $f$, then $\{Tf_n\}$ converges weakly to $Tf$.

*Solution.* $(a)$ If $f_n \xrightarrow{\|\cdot\|} f$, then for all $g \in \mathcal{H}$ we have $\langle f_n - f, g\rangle \leqslant \|f_n - f\| \, \|g\|$ converges to 0 making $f_n$ to converge to $f$ weakly. Similarly, the continuity of the norm guarantees that $\lim\limits_n \|f_n\| = \left\|\lim\limits_n f_n\right\| = \|f\|$.
Now we prove the converse.

Assume that we have $\lim_n \|f_n\| = \|f\|$ and $f_n \rightharpoonup f$, respectively.

It follows

$$\|f_n - f\|^2 = \langle f_n - f, f_n - f\rangle = \|f_n\|^2 + \|f\|^2 - \langle f_n, f\rangle - \langle f, f_n\rangle \to \|f\|^2 + \|f\|^2 - \|f\|^2 - \|f\|^2 = 0\,, \text{as}\, n \to \infty.$$

Hence $\lim_n \|f_n - f\| = 0$, -i.e., $f_n$ converges (strongly) to $f$.

$(b)$ Assume $\{f_n\}_{n\geqslant 1} \subseteq \mathcal{H}$ converges weakly to $f \in \mathcal{H}$. Let $g \in \mathcal{H}$ be given. Define $\ell : \mathcal{H} \to \mathbb{C}$ by $\ell(f) = \langle Tf, g\rangle$. It is immediate to see $\ell$ is linear. It is also bounded. Indeed, the Cauchy-Schwarz inequality provides for any $h \in \mathcal{H}$, $|\ell(h)| = |\langle Th, g\rangle| \leqslant \|Th\| \, \|g\| \leqslant \|T\| \, \|g\| \, \|h\|$. Then $\|\ell\| \leqslant \|T\| \, \|g\| < \infty$ because $T$ is bounded. Thence $\ell$ is a bounded linear functional. Riesz representation theorem therefore guarantees that there must exist $\xi \in \mathcal{H}$ such that $\ell = \langle . \, , \xi\rangle$. By weak convergence, it follows $\ell(f_n) = \langle f_n, \xi\rangle \to \langle f, \xi\rangle = \ell(f)$, -i.e., $\lim\limits_n \langle Tf_n, g\rangle = \langle Tf, g\rangle$ as desired.

# Problem 4

Let $\mathcal{H}$ be a Hilber space and let $T_n : \mathcal{H} \to \mathcal{H}$ be a sequence of bounded linear operators on $\mathcal{H}$ with $\|T_n\| \leqslant 1$ for all $n \in \mathbb{N}$. Suppose that for every vector $x \in \mathcal{H}$ the following holds:

$$T_i^* T_j x = 0,$$

for all $i, j \in \mathbb{N}$ with $i \neq j$.

(1) Prove that for every $i, j \in \mathbb{N}$, the ranges of $T_i$ and $T_j$ are orthogonal.

(2) Prove that for every $x \in \mathcal{H}$ the sequence $\{T_n x\}_{n\geqslant 1}$ is a Cauchy sequence.

(3) Let $T : \mathcal{H} \to \mathcal{H}$ be defined by $Tx = \lim_n T_n x$. Prove that $T$ is bounded and $\|T\| \leqslant 1$.

*Solution.* (1) For all $x, y \in \mathcal{H}$, we have $\langle T_i x, T_j y\rangle = \langle x, T_i^* T_j y\rangle$. Whenever $i \neq j$, $T_i^* T_j y = 0$, hence $\langle T_i x, T_j y\rangle = 0$.

(2) By orthogonality, for all $N \in \mathbb{N}$:

$$\sum_{n=1}^{N} \|T_n x\|^2 = \left\|\sum_{n=1}^{N} T_n x\right\|^2 \leqslant \left\|\sum_{n=1}^{N} T_n x\right\|^2 + \left\|x - \sum_{n=1}^{N} T_n x\right\|^2 = \|x\|^2 \,.$$

$x$ being fixed, this actually means that $\sum\limits_{n=1}^{N} \|T_n x\|^2 < \infty$. Hence $\lim\limits_n \|T_n x\|^2 = 0$. Thus for all $(m, n) \in \mathbb{N}^2$:

$$\|T_n x - T_m x\|^2 = \|T_n x\|^2 + \|T_m x\|^2 \to 0\,, \quad \text{as}\, m, n \to \infty.$$

$(c)$ Let $x \in H$ be given. For all $n \in \mathbb{N}$, $\|T_n x\| \leqslant \|T_n\| \|x\| \leqslant \|x\|$. Then $\lim_n \|T_n x\| \leqslant \|x\|$. By continuity, we have $\lim_n \|T_n x\| = \|\lim_n T_n x\| = \|Tx\|$. Hence $\|Tx\| \leqslant \|x\|$ which implies $\|T\| \leqslant 1$ therefore $T$ is proved to be bounded as desired.

# Problem 5

Consider the real line $\mathbb{R}$ equipped with the usual Euclidean metric.

$(a)$ Prove that if $A, B \subseteq \mathbb{R}$ are disjoint closed sets, then there exist disjoint open sets $U, V \subseteq \mathbb{R}$ such that $A \subseteq U$ and $B \subseteq V$.

$(b)$ Let $m^*$ denote the Lebesgue outer measure on $\mathbb{R}$. Prove that for any two sets $A, B \subseteq \mathbb{R}$ such that $\inf_{a \in A, b \in B} |a - b| > 0$ we have

$$m^*(A \cup B) = m^*(A) + m^*(B)\,.$$

*Solution.* $(a)$ Thanks to Rhoklin, we can construct the following sets:

$$U := \{x \in \mathbb{R} : \operatorname{dist}(x, A) < \operatorname{dist}(x, B)\}\ , \quad \text{and}\, V = \{x \in \mathbb{R} : \operatorname{dist}(x, A) > \operatorname{dist}(x, B)\}$$

The functions $x \mapsto d(x, A)$ and $x \mapsto d(x, B)$ being continuous, the sets $U$ and $V$ are open sets. Clearly we have $A \subseteq U$ and $B \subseteq V$.

PS: $d(x, C) := \inf_{c \in C} |x - c|$.

$(b)$ Let $A, B \subseteq \mathbb{R}$ be any subsets of $\mathbb{R}$ satisfying the hypothesis, -i.e., $d(A, B) > 0$. We shall use $\mathcal{B}(\mathbb{R})$ for the Borel $\sigma -$ algebdra on $\mathbb{R}$. Then

$$m^*(K) = \inf\{m(I) : I \in \mathcal{B}(\mathbb{R})\ \text{and}\, I \supseteq K\}$$

where $m(I)$ indicates the length of $I$.

**Recall.** A map $\lambda : \mathcal{P}(\Omega) \to [0, \infty]$ is called an *outer measure* on $\mathcal{P}(\Omega)$ if it satisfies the following conditions: $(\alpha)$ $\lambda(\emptyset) = 0$; $(\alpha\alpha)$ If $A \subseteq B$ then $\lambda(A) \leqslant \lambda(B)$, -i.e., $\lambda \geqslant 0$; $(\alpha\alpha\alpha)$ if $\{A_n\}_{n \in \mathbb{N}} \subseteq \mathcal{P}(\Omega)$, then for any $A \in \mathcal{P}(\Omega)$ such that $A \subseteq \bigcup_{n \in \mathbb{N}}$, we have $\lambda(A) \leqslant \sum_{n=1}^{\infty} \lambda(A_n)$. This last condition is equivalent to $\lambda\left(\bigcup_n A_n\right) \leqslant \sum_{n=1}^{\infty} \lambda(A_n)$.

Let $\mathcal{A}$ be an algebra and $\mu$ a measure, -i.e., a map which is $\sigma$*-additive* and *non-negative* on $\mathcal{A}$. The *outer measure associated to* $\mu$ is the map $\mu^* : \mathcal{P}(\Omega) \to [0, \infty]$, by:

$$\mu^*(A) := \inf\left\{\sum_{n \in \mathbb{N}} \mu(A) : A \subseteq \cup_{n \in \mathbb{N}} A_n\, \text{and}\ \{A_n\}_{n \geqslant 1} \in \mathcal{R}(A)\right\} \tag{$*$}$$

where $\mathcal{R}(A)$ indicates the set of all the countable covers of $A$ by elements taken from $\mathcal{A}$.

If $\mathcal{A}$ was actually a $\sigma-$algebra, then $(*)$ becomes $\mu^*(A) = \inf\{\mu(B) : B \in \mathcal{A}\, \text{and}\, B \supseteq A\}$. One could also define $\mu_*(A) = \sup\{\mu(B) : B \in \mathcal{A}\, \text{and}\, B \subseteq A\}$, and expect to have $\mu_*(A) = \mu^*(A)$ for any measurable set in the sense of the following definition.

Let $\lambda$ be an outer measure on $\Omega$. A subset $A \subseteq \Omega$ is said to be $\lambda-$**measurable** if, for any $B \subseteq \Omega$, $\lambda(B) = \lambda(B \cap A) + \lambda(B \cap A^c)$.

Since $m^*$ is an outer measure, it is $\sigma-$subadditive. Then $m^*(A \cup B) \leqslant m^*(A) + m^*(B)$. Consequently, we shall only prove that $m^*(A \cup B) \geqslant m^*(A) + m^*(B)$. Let $\delta > 0$ be given. There exists $I \in \mathcal{B}(\mathbb{R})$, the Borel $\sigma-$algebra, $A \cup B \subseteq I$ such that

$$m^*(I) + \delta \geqslant m(I) \geqslant m^*(A \cup B).$$

We claim $d\left(\overline{A}, \overline{B}\right) > 0$, where $\overline{S}$ is the closure of $S$ for any given set $S$. Let $(a,b) \in \overline{A} \times \overline{B}$ be arbitrary. There exists $(a_k, b_k) \in A \times B$ such that $(a,b) = \lim_k (a_k, b_k)$. For all $k \in \mathbb{N}$, $d(a_k, b_k) \geqslant d(A,B) > 0$. Then $d(a,b) = \lim_k d(a_k, b_k) \geqslant d(A,B) > 0$. Therefore $d\left(\overline{A}, \overline{B}\right) = \inf_{(a,b)\in\overline{A}\times\overline{B}} d(a,b) \geqslant d(A,B) > 0$. Since $\left(\overline{A} \cup \overline{B}\right) \cap I \subseteq I$, and since $\overline{A}, \overline{B} \in \mathcal{B}(\mathbb{R})$, we have

$$m\left(\left(\overline{A} \cap I\right) \cup \left(\overline{B} \cap I\right)\right) = m^*\left(\left(\overline{A} \cup \overline{B}\right) \cap I\right) \leqslant m(I) < m^*(A \cup B) + \delta$$

Since $m$ is additive(it's a measure), we have

$$m\left(\overline{A} \cap I\right) + m\left(\overline{B} \cap I\right) < m^*(A \cup B) + \delta$$

Moreover $A \subseteq \overline{A} \cap I$ and $B \subseteq \overline{B} \cap I$. Then $m^*(A) + m^*(B) \leqslant m\left(\overline{A} \cap I\right) + m\left(\overline{B} \cap I\right) < m^*(A \cup B) + \delta$. Letting $\delta \downarrow 0$, we obtain $m^*(A) + m^*(B) \leqslant m^*(A \cup B)$ as desired.

# Problem 6

Assume that you know the function

$$f_n(x) = \int_0^n t^{x-1}\left(1 - \frac{t}{n}\right)^n dt$$

is well defined for all $x > 0$ and $n \in \mathbb{N}$.
$(a)$ Apply a convergence theorem to show that $\lim_{n\to\infty} f_n(x)$ exists for all $x > 0$.
$(b)$ Write down the full statement of the convergence theorem you used in $(a)$.

*Solution.* $(a)$ For all $x > 0$ and for each $n \in \mathbb{N}$, we have

$$f_n(x) = \int_{(0,\infty)} t^{x-1}\left(1 - \frac{t}{n}\right)^n \mathbb{1}_{[0,n]}(t)\, d\lambda(t)$$

where $\lambda$ designates the usual Lebesgue measure on $\mathbb{R}$. Define $g_{n,x}(t) := t^{x-1}\left(1 - \frac{t}{n}\right)^n \mathbb{1}_{[0,n]}(t)$, for all $t > 0$. We have for all $t > 0$

$$\begin{aligned} |g_{n,x}(t)| &= \left| t^{x-1}\left(1 - \frac{t}{n}\right)^n \mathbb{1}_{[0,n]}(t) \right| \\ &= t^{x-1}\left(1 - \frac{t}{n}\right)^n \mathbb{1}_{[0,n]}(t) \\ &= t^{x-1} e^{n \ln\left(1 - \frac{t}{n}\right)} \mathbb{1}_{[0,n]}(t) \end{aligned}$$

Notice that for all $0 < \xi < 1$, $\ln(1-\xi) \leqslant -\xi$.(This is true even when $\xi \leqslant 0$.) Then

$$|g_{n,x}(t)| \leqslant t^{x-1} e^{n\left(-\frac{t}{n}\right)} \mathbb{1}_{[0,n]}(t) \leqslant t^{x-1} e^{-t}.$$

Define $h_x(t) := t^{x-1}e^{-t}$ for all $t > 0$ which is indepedent of $n$. We claim that $h_x$ is integrable for all $x > 0$. Indeed $h_x$ is integrable on each compact $[\alpha, \beta] \subset (0, \infty)$ where $0 < \alpha \leqslant \beta < \infty$ since

*Case1*: $0 < x < 1$

$$0 \leqslant \beta^{x-1} e^{-\beta} (\beta - \alpha) \leqslant \int_{[\alpha,\beta]} t^{x-1} e^{-t} d\lambda(t) \leqslant \alpha^{x-1} e^{-\alpha} < \infty.$$

*Case2*: $x \geqslant 1$

$$0 \leqslant \alpha^{x-1} e^{\beta} \leqslant \int_{[\alpha,\beta]} t^{x-1} e^{-t} d\lambda(t) \leqslant \beta^{x-1} e^{-\alpha} < \infty.$$

We therefore must only study the convergence of the integral at 0 and $\infty$. First, in any neighborhood $(0, \epsilon]$ of 0, where $\epsilon > 0$ is viwed as small, we have $h_x(t) = o(t^{x-1})$. Then $\int_{(0,\epsilon]} t^{x-1} dt = o\left(\int_{(0,\epsilon]} t^{x-1} dt\right)$. But, $\int_{(0,\epsilon]} t^{x-1} dt = \lim_{0<\delta<\epsilon, \delta\downarrow 0} \int_{[\delta,\epsilon]} t^{x-1} dt = \lim_{0<\delta<\epsilon, \delta\downarrow 0} \left(\frac{\epsilon^x}{x} - \frac{\delta^x}{x}\right) = \frac{\epsilon^x}{x} < \infty$. Hence, $h_x$ is integrable in any neighborhood of $t = 0$. In the same vein, at any neighborhood $[M, \infty\ )$ of $\infty$, where $M > 0$ is viwed as large, we have $h_x(t) = o\left(e^{-\frac{t}{2}}\right)$. But we know $0 \leqslant \int_{[M,\infty)} e^{-\frac{t}{2}} dt = 2e^{-\frac{M}{2}} < \infty$ making $e^{-t}$ integrable. Hence, $h_x$ is integrable in any neighborhood of $\infty$. These facts allow the conclusion that $h_x$ is integrable in $(0, \infty)$. Additionally, we check that $g_{n,x}(t) \to t^{x-1}e^{-t}$ as $n \to \infty$ pointwisely. To see this, notice that for all $t > 0$, there exists $n_0 \in \mathbb{N}$ so that $n_0 \geqslant t$. Then for all $n \geqslant n_0$(in particular, $n$ does depend on $t$):

$$\left|g_{n,x}(t) - t^{x-1}e^{-t}\right| = t^{x-1} \left|e^{n \ln\left(1-\frac{t}{n}\right)} \mathbb{1}_{[0,n]}(t) - e^{-t}\right| = t^{x-1} \left|e^{n \ln\left(1-\frac{t}{n}\right)} - e^{-t}\right| \to 0\,, \text{as}\, n \to \infty.$$

We can therefore apply the Lebesgue Dominated Convergence Theorem to conclude that

$$\lim_{n\to\infty} f_n(x) = \int_{(0,\infty)} \lim_{n\to\infty} g_{n,x}(t)\, d\lambda(t) = \int_{(0,\infty)} t^{x-1} e^{-t} d\lambda(t) = \int_0^\infty t^{x-1} e^{-t} dt = \Gamma(x)\,.$$

$(b)$ Statement of LDCT(cf. Edwin Hewitt and Karl Stromberg)

Let the measured space $(X, \mathcal{A}, \mu)$ be given.

*Statement of Lebesgue Dominated Convergence Theorem.* Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of extended real-value, $\mathcal{A}-$measurable functions each defined a.e. on $X$, and suppose there exists a function $h$ in $L^1(X)$ such that for each integer $n$, the inequality $|f_n(x)| \leqslant h(x)$ holds a.e. on $X$. Then

$$(i) \quad \int_X \underline{\lim}_n f_n d\mu \leqslant \underline{\lim}_n \int_X f_n d\mu; \text{ and}$$

$$(ii) \quad \int_X \overline{\lim}_n f_n d\mu \geqslant \overline{\lim}_n \int_X f_n d\mu.$$

Put differently,

$$\int_X \underline{\lim}_n f_n d\mu \leqslant \underline{\lim}_n \int_X f_n d\mu \leqslant \overline{\lim}_n \int_X f_n d\mu \leqslant \int_X \overline{\lim}_n f_n d\mu.$$

If additionally $\lim_n f_n(x)$ exists for $\mu-$almost every $x$ on $X$, [1] then $\displaystyle\int_X \lim_n f_n d\mu$ exists and

$$(iii) \quad \lim_n \int_X f_n d\mu = \int_X \lim_n f_n d\mu.$$

# Problem 7

Let $1 \leqslant p < \infty$ and $f \in L^p(\mathbb{R})$. Denote by $m$ the Lebesgue measure on $\mathbb{R}$.

(a) Prove that $lim_{n\to\infty} m[\{x \in \mathbb{R} : |f(x)| \geqslant n\}] = 0$.

(b) Prove that the set $\{x \in \mathbb{R} : |f(x)| \neq 0\}$ is $\sigma-$finite.

(c) Prove that $m[\{x \in \mathbb{R} : |f(x)| = \infty\}] = 0$.

*Solution.* $(a)$ We have for each integer $n$, $\displaystyle\int_{\mathbb{R}} |f|^p\, dm = \int_{\{|f|\geqslant n\}} |f|^p\, dm + \int_{\{|f|<n\}} |f|^p\, dm \geqslant \int_{\{|f|\geqslant n\}} |f|^p\, dm \geqslant n^p m[\{|f| \geqslant n\}]$. Then $\displaystyle m[\{|f| \geqslant n\}] \leqslant \frac{1}{n^p}\int_{\mathbb{R}} |f|^p\, dm$. Since $\displaystyle\int_{\mathbb{R}} |f|^p\, dm < \infty$, the bound $\displaystyle\frac{1}{n^p}\int_{\mathbb{R}} |f|^p\, dm$ goes to 0. Hence $\lim_n m[\{|f| \geqslant n\}] = 0$.

$(b)$ We have $\displaystyle\{|f| \neq 0\} = \bigcup_{n=1}^{\infty} \{n-1 < |f| \leqslant n\}$, and for each $n$, $m[\{n-1 < |f| \leqslant n\}] = m[\{|f| > n-1\}] - m[\{|f| > n\}]$ is finite since for each positive integer $k$, $0 \leqslant m[\{|f| > k\}] \leqslant m[\{|f| \geqslant k-1\}] < \infty$. Notice that $m[\{|f| \geqslant n\}] \to 0$ implies $\{|f| \geqslant n\}$ is bounded, therefore finite.

$(c)$ $\{|f| = \infty\} = \bigcap_{n\geqslant 1} A_n$ wherer $A_n := \{|f| \geqslant n\}$ which satisfies $A_n \supset A_{n+1} \supset \ldots$. Since for all positive integers $m(\{|f| \geqslant n\}) < \infty$, it follows that $m[\{|f| = \infty\}] = \lim_n m[A_n] = \lim_n m[\{|f| \geqslant n\}] = 0$.

[1]his means $f_n$ converges pointwisely $\mu-$almost everywhere on $X$.

# Problem 8

Denote by $m$ the Lebesgue measure on $\mathbb{R}$. Let $E \subset \mathbb{R}$ be a measurable set with $m(E) < \infty$. Suppose $F : E \to \mathbb{R}$ is a measurable function, so that $f(x) > 0$ for a.e. $x \in E$. Prove that if $\{E_n\}$ is a sequence of measurable subsets of $E$, so that

$$\lim_n \int_{E_n} f(x)dx = 0$$

then $\lim_{n\to\infty} m(E_n) = 0$.

*Solution.* Consider $B = \{x \in E : f(x) > 0\}$. Since $B$ has $E$'s full measure, for each positive integer $n$, $\displaystyle\int_{E_n} fdm = \int_{E_n\cap B} fdm = \int_{\mathbb{R}} \mathbb{1}_{E_n\cap B} fdm$. By Fatou's lemma,

$$0 \leqslant \int_{\mathbb{R}} \varliminf_n \mathbb{1}_{E_n\cap B} fdm \leqslant \varliminf_n \int_{\mathbb{R}} \mathbb{1}_{E_n\cap B} fdm = 0.$$

Then $\int_{\mathbb{R}} \varliminf_n \mathbb{1}_{E_n\cap B} fdm = 0$. Using the fact $f > 0$ on $B$, we actually have $\varliminf_n \mathbb{1}_{E_n\cap B} f = 0$ which implies $\varliminf_n \mathbb{1}_{E_n\cap B} = 0$. Now for each positive integer $n$,

$$\begin{aligned}
\sup_{k\geqslant n} \mathbb{1}_{E_k\cap B} \leqslant \mathbb{1}_{\bigcup_{k=n}^{\infty}(E_k\cap B)} &\implies \varlimsup_n \mathbb{1}_{E_n\cap B} \leqslant \lim_{n\to\infty} \mathbb{1}_{\bigcup_{k\geqslant n}(E_k\cap B)} \\
&\implies \int_{\mathbb{R}} \varlimsup_n \mathbb{1}_{E_n\cap B} dm \leqslant \int_{\mathbb{R}} \lim_{n\to\infty} \mathbb{1}_{\bigcup_{k\geqslant n}(E_k\cap B)} dm.
\end{aligned}$$

Additionally, the sequence $\left\{\mathbb{1}_{\bigcup_{k\geqslant n}(E_k\cap B)}\right\}_{n=1}^{\infty}$ is a sequence of real valued measurable functions each defined on $\mathbb{R}$ , such that for every positive integer $n$, $\left|\mathbb{1}_{\bigcup_{k\geqslant n}(E_k\cap B)}(x)\right| \leqslant \mathbb{1}_E(x)$ for all $x \in \mathbb{R}$ with $\displaystyle\int_{\mathbb{R}} \mathbb{1}_E dm = m(E) < \infty$. Then, all the conditions for the application of LDCT are met, and therefore

$$\int_{\mathbb{R}} \lim_{n\to\infty} \mathbb{1}_{\bigcup_{k\geqslant n}(E_k\cap B)} = \lim_{n\to\infty} \int_{\mathbb{R}} \mathbb{1}_{\bigcup_{k\geqslant n}(E_k\cap B)} = \lim_{n\to\infty} m\left(\bigcup_{k\geqslant n}(E_k\cap B)\right).$$

Moreover $\bigcup_{k\geqslant n}(E_k\cap B) = \bigcup_{k=1}^{\infty}(E_k\cap B) \setminus \bigcup_{k=1}^{n-1}(E_k\cap B)$. Then

$$\begin{aligned}
\lim_{n\to\infty} m\left(\bigcup_{k\geqslant n}(E_k\cap B)\right) &= \lim_n \left[m\left(\bigcup_{k=1}^{\infty}(E_k\cap B)\right) - m\left(\bigcup_{k=1}^{n-1}(E_k\cap B)\right)\right] \\
&= m\left(\bigcup_{k=1}^{\infty}(E_k\cap B)\right) - m\left(\bigcup_{k=1}^{\infty}(E_k\cap B)\right) \\
&= 0.
\end{aligned}$$

Hence $\varlimsup_n \mathbb{1}_{E_k\cap B} = 0$ a.e. Consequently, $\lim_n \mathbb{1}_{E_n\cap B} = 0$ a.e. Then by dominated convergence, we have

$$\lim_n m\left(E_k \cap B\right) = \lim_n \int_{\mathbb{R}} \mathbb{1}_{E_n \cap B} dm = \int_{\mathbb{R}} \lim_n \mathbb{1}_{E_n \cap B} dm = 0.$$

Finally notice that $m\left(E_n\right) = m\left(E_n \cap B\right) + m\left(E_n \cap \left(E \setminus B\right)\right) = m\left(E_n \cap B\right)$. Therefore $\lim_n m\left(E_n\right) = 0$.