Analysis, Clemson Preliminary Exam: January 2024

Hamilijaona Elmar

Problem 1

Let (X, d) be a metric space and let $\{K_n\}_{n\geqslant 1}$ be a sequence of compact sets.

(a) Prove or disprove: $\bigcap_{n\geqslant 1}$ is compact.
(b) Prove or disprove: $\bigcup_{n\geqslant 1}$ is compact.

Solution. (a) Plainly, K is closed, hence *complete* because as X is compact, it is also complete. Now we prove that K is totally bounded. Let $\epsilon > 0$ be arbitrary. For each $n \in \mathbb{N}$ there exists $\{x_i^n\}_{i=1}^N \subset K_n$, $N \in \mathbb{N}$ so that $\bigcup_{1\leqslant i\leqslant N} B_{\frac{\epsilon}{2}}(x_i^n) \supset K_n$. We have that $\bigcap_{n\geqslant 1}\bigcup_{1\leqslant i\leqslant N} B_{\frac{\epsilon}{2}}(x_i^n) \supset K$. Since $\bigcap_{n\geqslant 1}\bigcup_{1\leqslant i\leqslant N} B_{\frac{\epsilon}{2}}(x_i^n) = \bigcup_{1\leqslant i\leqslant N}\bigcap_{n\geqslant 1} B_{\frac{\epsilon}{2}}(x_i^n)$, then given any $y \in K$, there exists a positive integer $1 \leqslant i_y \leqslant N$ so that for all $n \geqslant 1$, $y \in B_{\frac{\epsilon}{2}}\left(x_{i_y}^n\right)$. Put differently, for all $n \geqslant 1$, we have $x_{i_y}^n \in B_{\frac{\epsilon}{2}}(y)$. Notice that $a \in B_\delta(b)$ if and only if $d(a, b) < \delta$ if and only if $b \in B_\delta(a)$ which implies $B_{2\delta}(b) \supset B_\delta(a)$. Thus we have for each positive integer n, $B_{\frac{\epsilon}{2}}\left(x_{i_y}^n\right) \subset B_\epsilon(y)$. In other words, for each $1 \leqslant j \leqslant$, there exists $y_j \in K$ so that $B_{\frac{\epsilon}{2}}\left(x_j^n\right) \subset B_\epsilon(y_j)$ for all $n \in \mathbb{N}$. Then $\bigcap_{n\geqslant 1} B_{\frac{\epsilon}{2}}\left(x_j^n\right) \subset B_\epsilon(y_j)$. Hence we found $\{y_j\}_{1\leqslant j\leqslant N} \subset K$ satisfying $\bigcup_{j=1}^N B_\epsilon(y_j) \supset \bigcup_{j=1}^N N_{\frac{\epsilon}{2}}\left(x_j^n\right) \supset K$. Hence K is totally bounded, consequently it is compact.

(b) $X = [0, 1]$, $K_n = \left[0, 1 - \frac{1}{n}\right]$. We have X is compact and, for each positive integer $n \in \mathbb{N}$, K_n is compact. Yet $\bigcup_{n\geqslant 1} K_n = [0, 1)$ is not compact because it is not closed.

Problem 2

Consider the Banach space $C([0, 1])$ consisting of all continous functions $f : [0, 1] \to \mathbb{R}$, equipped with the supremum norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Consider the operator $T : C([0, 1]) \to C([0, 1])$ given by

$$Tf(x) = \int_0^1 e^{xy} f(y)\, dy.$$

(a) Prove that T is bounded.
Compute $\|T\|$. Justify your answer.

Solution. (a) For all $f \in C([0, 1])$ and for all $x \in [0, 1]$, we have

$$|Tf(x)| = \left|\int_0^1 e^{xy} f(y)\, dy\right| \leqslant \int_0^1 e^{xy} |f(y)|\, dy \leqslant \left(\int_0^1 e^{xy} dy\right) \|f\|_\infty .$$

Then $\|Tf\|_\infty \leqslant \left(\sup_{x\in[0,1]} \int_0^1 e^{xy} dy\right) \|f\|_\infty \leqslant \left(\int_0^1 \sup_{x\in[0,1]} e^{xy} dy\right) \|f\|_\infty = \left(\int_0^\infty e^{y} dy\right) \|f\|_\infty = (e-1)\|f\|_\infty$. Hence T bounded with $\|T\| \leqslant e-1$.

(b) We have for all $x \in [0,1]$, $|T1(x)| = \int_0^1 e^{xy} dy = \dfrac{e^x - 1}{x} =: F(x)$. F is differentiable in $(0,1)$ and we have that $F'(x) = \frac{(x-1)e^x+1}{x^2} = \frac{1}{x}\left(\frac{(x-1)e^x+1}{x}\right) = \frac{\theta e^\theta}{x} > 0$ for some $0 < \theta < x$. Hence F is monotonically increasing. Then for all $x \in [0,1]$, we have $F(x) \leqslant F(1) = e-1$. Then $\sup_{x\in[0,1]} F(x) = e-1$. Hence $\|T1\|_\infty = \sup_{x\in[0,1]} |T1(x)| = e-1$ which implies $\|T\| = \sup_{f\neq 0} \frac{\|Tf\|_\infty}{\|f\|_\infty} \geqslant \frac{\|T1\|_\infty}{\|1\|_\infty} = e-1$.

Problem 3

Let $\{x_n\}$ be a sequence in a normed linear space(n.l.s) $(X, \|\cdot\|)$ and $X \in X$.

(a) Prove that if $x_n \to x$, then $\frac{x_1+\cdots+x_n}{n} \to x$.
(b) It is not always true that if $\frac{x_1+\cdots+x_n}{n} \to x$, then $x_n \to x$. Provide a counterexample.
(c) Prove that if $\frac{x_1+\cdots+x_n}{n} \to x$, then $\frac{x_n}{n} \to 0$.

Solution. (a) Let $\epsilon > 0$ be given. There exists $N \in \mathbb{N}$ so that $\|x_n - x\| < \epsilon$ whenever $n \geqslant N$. Then for all $n \geqslant N$, we have

$$\begin{aligned}
\left\|\frac{1}{n}\sum_{i=1}^n x_i - x\right\| = \left\|\frac{1}{n}\sum_{i=1}^n (x_i - x)\right\| &\leqslant \left\|\frac{1}{n}\sum_{i=1}^{N-1} (x_i - x)\right\| + \left\|\frac{1}{n}\sum_{i=N}^{N} (x_i - x)\right\| \\
&\leqslant \frac{1}{n}\left\|\sum_{i=1}^{N-1} (x_i - x)\right\| + \frac{1}{n}\sum_{i=N}^{n} \|x_i - x\| \\
&< \frac{1}{n}\left\|\sum_{i=1}^{N-1} (x_i - x)\right\| + \frac{(n-N+1)\epsilon}{n} < \frac{1}{n}\left\|\sum_{i=1}^{N-1} (x_i - x)\right\| + \epsilon.
\end{aligned}$$

But since N is fixed, it follows that $\left\|\sum_{i=1}^{N-1} (x_i - x)\right\| < n\epsilon$ whenever $n \geqslant N'$ for some $N' \in \mathbb{N}$. Therefore for all $n \geqslant \max\{N, N'\}$,

$$\left\|\frac{1}{n}\sum_{i=1}^n x_i - x\right\| < \epsilon + \epsilon = 2\epsilon.$$

In addition, $\epsilon > 0$ was arbitrary. This implies that $\lim_{n\to\infty} \left\|\frac{1}{n}\sum_{i=1}^n x_i - x\right\| = 0$.

(b) It suffices to find a sequence $\{x_n\}_{n\geqslant}$ such that $\sum_{i=1}^n x_i$ grows slower that n. For example, it is enough to find $\{x_n\}_{n\geqslant 1}$ with $\sum_i^n x_i = O(\log n)$. Consider the sequence of real numbers $\{x_n\}_{n\geqslant 1}$ defined for each positive integer n by

$$x_n = \begin{cases} 1 & \text{if}\, n = 2^k\,, \text{for some}\, k \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}$$

We have for each $n \in \mathbb{N}$, $\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{\text{number of 1's}}{n}$. If $2^k \leqslant x_n < 2^{k+1}$, the number of 1's is $k = \log_2\left(2^k\right) = O\left(\log_2(n)\right)$. Then,

$$\frac{\sum_{i=1}^{n} x_i}{n} = O\left(\frac{\log_2(n)}{n}\right).$$

Since $\lim_{n\to\infty} \frac{\log_2(n)}{n} = 0$, it follows that $\lim_{n\to\infty} \frac{\sum_{i=1}^{n} x_i}{n} = 0$.

Another example exploits the compensation phenomenon. Consider $x_n = (-1)^n$. Due to compensation, we have for each $n \in \mathbb{N}$,

$\dfrac{x_1 + \cdots + x_n}{n} = 0$ or 1 according to wether n is an even integer or an odd. In any case, we have that $\frac{x_1+\cdots+x_n}{n} \to 0$ when $n \to \infty$.

(c) Assume $\lim_{n\to\infty} \left\|\frac{1}{n}\sum_{i=1}^{n} x_i - x\right\| = 0$. Then we have for each $n \in \mathbb{N}$,

$$\begin{aligned} \left\|\frac{x_n}{n}\right\| = \left\|\frac{1}{n}\sum_{i=1}^{n} - \frac{1}{n}\sum_{i=1}^{n-1} x_i\right\| &\leqslant \left\|\frac{1}{n}\sum_{i=1}^{n} x_i - x\right\| + \left\|\frac{1}{n}\sum_{i=1}^{n-1} x_i - x\right\| \\ &= \left\|\frac{1}{n}\sum_{i=1}^{n} x_i - x\right\| + \frac{n-1}{n}\left\|\frac{1}{n-1}\sum_{i=1}^{n-1} x_i - \frac{n}{n-1}x\right\| \\ &\leqslant \left\|\frac{1}{n}\sum_{i=1}^{n} x_i - x\right\| + \frac{n-1}{n}\left\|\frac{1}{n-1}\sum_{i=1}^{n-1} x_i - x\right\| + \frac{n-1}{n}\left\|x - \frac{n}{n-1}x\right\| \\ &= \left\|\frac{1}{n}\sum_{i=1}^{n} x_i - x\right\| + \frac{n-1}{n}\left\|\frac{1}{n-1}\sum_{i=1}^{n-1} x_i - x\right\| + \frac{1}{n}\|x\| \end{aligned}$$

Since each of the last three terms goes to 0, we conclude that $\left\|\frac{x_n}{n}\right\| \to 0$ as $n \to \infty$.

Problem 4

Let X be the set of all real sequences $\{x_n\}$ with finitely many non-zero terms, i.e., $x_n \neq 0$ only for a finite number of $n \in \mathbb{N}$. Define an inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ by

(1) Prove or disprove: $(X, \langle \cdot, \cdot \rangle)$ is separable.
(2) Prove or disprove: $(X, \langle \cdot, \cdot \rangle)$ is complete.

Solution. (1) Let $D = \left\{\{q_n\}_{n\geqslant 1} : q_n \in \mathbb{Q}\, \text{and}\, q_n \neq 0\, \text{only for finitely many}\, q_n\right\}$. Plainly we have that $D \sim \bigcup_{N\in\mathbb{N}} \mathbb{Q}^N$(equipotent), therefore D is countable. Let $\{x_n\}_{n\geqslant 1} \subset X$. Let $\epsilon > 0$ be arbitrary. Since $x_n \neq 0$ only for finitely many n's, it follows that $x_n = 0$ for all $n > N$ for some $N \in \mathbb{N}$. For each $1 \leqslant i \leqslant N$, there exists $q_i \in \mathbb{Q}$ so that $|x_i - q_i| < \frac{\epsilon}{\sqrt{N}}$. This results from the denseness of $\mathbb{Q}$ in $\mathbb{R}$. The sequence $q := \{q_1, \dots, q_N, 0, \dots\}$ belongs to D. It moreover satisfies,

$$\|x - q\| = \sqrt{\langle \{x_n\}, \{q_n\} \rangle} = \sqrt{\sum_{i=1}^{\infty} |x_i - q_i|^2} < \sqrt{\sum_{i=1}^{N} \frac{\epsilon^2}{N}} = \epsilon.$$

Hence D is both countable and dense proper subset of X, implying that X is separable.

(2) Consider the sequence $\{y_n\}$ defined for each $n \in \mathbb{N}$ by $y_n = \{x_i^n\}_{i \geqslant 1}$ where for each $i \geqslant 1$

$$x_i^n = \begin{cases} 2^{-i} & \text{when } i \leqslant n \\ 0 & \text{otherwise} \end{cases}$$

Plainly we have for each n, $y_n \in X$. For all $m > n \geqslant 1$,

$$\|y_m - y_n\| = \sqrt{\sum_{i=0}^{m} |x_i^m - x_i^n|^2} = \sqrt{\sum_{i=0}^{n} |2^{-i} - 2^{-i}|^2 + \sum_{i=0}^{n} |2^{-i}|^2} = \sqrt{\sum_{i=n}^{m} 2^{-2i}} \leqslant \sqrt{\sum_{i=n}^{\infty} 2^{-2i}}.$$

Since $\sum_{n \geqslant 1} 2^{-2n}$ is convergent, it must be true that $\sum_{i \geqslant n} 2^{-2i} \to 0$ when $n \to \infty$. Consequently, $\|y_m - y_n\| \to 0$ whenever $m, n \to \infty$. Hence $\{y_n\}$ is Cauchy. Yet, its limit which is $y := \{2^{-i}\}_{i=1}^{\infty}$ does not belong to X.

Problem 5

Let X be a inner product space.

(a) For $x, y \in X \setminus \{0\}$, define $\overline{x} = \frac{x}{\|x\|^2}$ and $\overline{y} = \frac{y}{\|y\|^2}$. Prove that

$$\|\overline{x} - \overline{y}\| = \frac{\|x - y\|}{\|x\| \, \|y\|}.$$

(b) Prove that $\|x - y\| \, \|z\| \leqslant \|x - z\| \, \|y\| + \|z - y\| \, \|x\|$ for all $x, y, z \in X$.

Solution. (a) Let φ be the inner product X is equipped by with. Then, given $x, y \in X \setminus \{0\}$, we have

$$\|\overline{x} - \overline{y}\|^2 = \varphi(\overline{x} - \overline{y}, \overline{x} - \overline{y}) = \|\overline{x}\|^2 + \|\overline{y}\|^2 - 2\varphi(\overline{x}, \overline{y}) = \frac{1}{\|x\|^2} + \frac{1}{\|y\|^2} - \frac{2}{\|x\|^2 \|y\|^2} \varphi(x, y).$$

But $2\varphi(x, y) = \|x\|^2 + \|y\|^2 - \|x - y\|^2$. Then

$$\|\overline{x} - \overline{y}\|^2 = \frac{1}{\|x\|^2} + \frac{1}{\|y\|^2} - \frac{1}{\|x\|^2 \|y\|^2} \left(\|x\|^2 + \|y\|^2 - \|x - y\|^2 \right) = \left(\frac{\|x - y\|}{\|x\| \, \|y\|} \right)^2.$$

Hence $\|\overline{x}-\overline{y}\| = \frac{\|x-y\|}{\|x\|\|y\|}$.

(b) Let $x, y, z \in \mathbb{N}$ be given. Define $\overline{x} = \frac{x}{\|x\|^2}$, $\overline{y} = \frac{y}{\|y\|^2}$ and $\overline{z} = \frac{z}{\|z\|^2}$. Using the previous result, we have

$$\begin{aligned}\|x-y\|\,\|z\| &= \|z\|\,\|x\|\,\|y\|\,\frac{\|x-y\|}{\|x\|\,\|y\|} = \|z\|\,\|x\|\,\|y\|\,\|\overline{x}-\overline{y}\| \leqslant \|z\|\,\|x\|\,\|y\|\left(\|\overline{x}-\overline{z}\| + \|\overline{y}-\overline{z}\|\right)\\ &= \|z\|\,\|x\|\,\|y\|\left(\frac{\|x-z\|}{\|x\|\,\|z\|} + \frac{\|y-z\|}{\|y\|\,\|z\|}\right) = \|x-z\|\,\|y\| + \|y-z\|\,\|x\|\,.\end{aligned}$$

Problem 6

Let $(X, \mathcal{F}, \mu)$ be a measure space. Show that if $A, B \in \mathcal{F}$ and $\mu(A\Delta B) = 0$ then $\mu(A) = \mu(B)$. Recall that $A\Delta B = (A \setminus B) \cup (B \setminus A)$.

Solution. We have $A = (A \cap B) \cup (A \setminus B)$ and $B = (B \cap A) \cup (B \setminus A)$. If $\mu(A\Delta B) = 0$, then by $\mu-$additivity $\mu(A \setminus B) + \mu(B \setminus A) = 0$, i.e., $\mu(A \setminus B) = 0 = \mu(B \setminus A)$. In addition, again by $\mu-$additivity, we have

$$\mu(A) = \mu(A \cap B) + \mu(A \setminus B) = \mu(A \cap B) = \mu(B \cap A) + \mu(B \setminus A) = \mu(B)\,.$$

Problem 7

Let $f \in L^2(0, \infty)$. Prove that

$$\lim_{n\to\infty} \int_0^\infty \frac{f(x)}{1+nx}dx = 0.$$

Solution. By Hölder's inequality, for each $n \in \mathbb{N}$,

$$\int_0^\infty \frac{f(x)}{1+nx}dx \leqslant \|f\|_{L^2(0,\infty)} \left(\int_0^\infty \frac{1}{(1+nx)^2}\right)^{\frac{1}{2}}$$

where $\|f\|_{L^2(0,\infty)}$ is the norm-2 of f which is finite because $f \in L^2(0,\infty)$. Now define for each $n \in \mathbb{N}$, $g_n(x) = \frac{1}{(1+nx)^2}$ for all $x > 0$. Then $\{g_n\}$ is a sequence of measurable functions, each defined everywhere on $(0, \infty)$ and $g_n \to 0$ pointwise on X. Additionally, for each $n \in \mathbb{N}$,

$$|g_n(x)| \leqslant \frac{1}{(1+x)^2} =: h(x)\ \text{, for all}\, x > 0,$$

with $\displaystyle\int_{(0,\infty)} h d\mu = \int_0^\infty \frac{1}{(1+x)^2}dx = \left(-\frac{1}{1+x}\right|_0^\infty = 1$, μ being the Lebesgue measure on $\mathbb{R}$. Lebesgue dominated convergence theorem therefore implies that

$$\lim_{n\to\infty}\int_0^\infty \frac{1}{(1+nx)^2}dx = \int_0^\infty \lim_{n\to\infty}\frac{1}{(1+nx)^2}dx = 0.$$

Consequently, this proves that $\lim_{n\to\infty}\int_0^\infty \frac{f(x)}{1+nx}dx = 0$.

Problem 8

Recall that a sequence $\{f_n\}$ of integrable functions converges in L^1 to an integrable function f if

$$\lim_{n\to\infty}\int_{\mathbb{R}} |f_n(x) - f(x)|\, d\mu(x) = 0.$$

Provide

(a) a sequence $\{f_n\}$ of integrable functions that converges uniformly to 0 function but does not converge in L^1.

(b) a sequence $\{f_n\}$ of function functions that converges in L^1 to the 0 function but does not converge pointwise.

Solution. (a) Consider $f_n = \frac{1}{n}\mathbb{1}_{[0,n]}$, and let μ denote the Lebesque measure on $\mathbb{R}$. For each $n \in \mathbb{N}$, we have $\|f_n\|_\infty \sup_{x\in\mathbb{R}} |f_n(x)| = \frac{1}{n}$. Then $\|f_n\|_\infty \to 0$ when $n \to \infty$ making $\{f_n\}$ to converge uniformly to the 0 function. Yet, for all $n \in \mathbb{N}$, $\int_{\mathbb{R}} \frac{1}{n}\mathbb{1}_{[0,n]} d\mu(x) = \frac{1}{n}\mu([0,n]) = 1 \not\to 0$ when $n \to \infty$.

(b) [Classic] $X = [0,1]$, $\mathcal{M} = \mathcal{B}([0,1])$ and μ =Lebesgue measure. For each n, define $f_n := \mathbb{1}_{J_n}$ where $J_n = \left[\frac{j}{2^{k+1}}, \frac{j+1}{2^{k+1}}\right]$ with $k \in \mathbb{N}$ is the unique non negative integer that satisfies $2^k \leqslant n < 2^{k+1}$ and $j := n - 2^k$. We have for each $n \in \mathbb{N}$,

$$\int_X |f_n(x)|\, d\mu(x) = \int_{[0,1]} \mathbb{1}_{J_n}(x)\, d\mu(x) = \mu(J_n) = \frac{1}{2^{k+1}} < \frac{1}{n}.$$

Now let $x \in [0,1]$ be given. For all $k \in \mathbb{N}$, $[0,1] = \bigcup_{\ell=0}^{2^{k+1}-1}\left[\frac{\ell}{2^{k+1}}, \frac{\ell+1}{2^{k+1}}\right] = \bigcup_{\ell=0}^{2^{k+1}-1} J_{2^k+\ell}$. Then $x \in J_{2^k+\ell_{x,k}}$ for some $0 \leqslant \ell_{x,k} < 2^{k+1}$. On the set $B := \bigcup_{k\geqslant 1} J_{2^k+\ell_{x,k}}$ we have $f_{2^i+\ell_{x,i}}(x) = 1 \not\to 0$ as $i \to \infty$. In addition,

$$\mu(B) \geqslant \mu\left(J_{2+\ell_{x,1}}\right) = \mu\left(\left[\frac{\ell_{x,1}}{2^2}, \frac{\ell_{x,1}+1}{2^2}\right]\right) = \frac{1}{4} > 0.$$

Problem 9

Let $\{f_n\}$ be a sequence of characteristic functions of measurable sets in $\mathbb{R}$ so that $\{f_n\}$ converges in L^1 . Prove that f coincide almost everywhere with a characteristic function of a measurable set.

Solution. Since for each $n \in \mathbb{N}$, f_n is a characteristic function, we have that $f_n(x) \in \{0,1\} \subseteq [0,1]$ and $[0,1]$ is compact. Since $f_n \to f$ pointwise, necessarily $f(x) \in [0,1]$ for all $x \in \mathbb{R}$. Assume there exists $B \in \mathcal{B}(\mathbb{R})$ such that $\mu(B) > 0$ and on which $f(x) \in (\epsilon, 1-\epsilon)$ for some $0 < \epsilon < 1$ (actually it is enough to take $0 < \epsilon \leqslant \frac{1}{2}$.) Then on B, we have

$$|f_n(x) - f(x)| > \epsilon.$$

Then it would follow that $\lim_{n\to\infty} |f_n(x) - f(x)| \geqslant \epsilon > 0$ on the measurable set B with positive measure, contradicting the fact that $f_n \to f$ pointwise a.e. Hence f must be a $\{0,1\}-$valued function a.e.. In other words, f must be $\mu-$a.e. the characteristic function of a measurable set.

Problem 10

Let μ^* be an outer measure on $\mathbb{R}$ with the property that for any two sets $U, V \subset \mathbb{R}$, if $d(U,V) > 0$ then $\mu^*(U \cup V) = \mu^*(U) + \mu^*(V)$. Prove that every Borel set is μ^* measurable.

Solution. Let $B \in \mathcal{B}(\mathbb{R})$ be a closed subset of $\mathbb{R}$. For each $n \in \mathbb{N}$, consider the sequence of subsets of $\mathbb{R}$ defined by $K_n = \left\{x \in \mathbb{R} : d(x,B) \geqslant \frac{1}{n}\right\}$. The set $A_n := (A \cap B) \cup (B \cap K_n)$ satisfies $\mu^*(A_n) = \mu^*(A \cap B) + \mu^*(B \cap K_n)$ because if $d(A \cap B, B \cap K_n) = 0$ there would exists $\{(a_k, b_k)\} \in (A \cap B) \times \overline{B \cap K_n} \subset B \times K_n$ such that $0 = \lim_{k\to\infty} d(a_k, b_k)$. But for each $k \in \mathbb{N}$, $d(a_k, b_k) \geqslant d(B, b_k) = d(b_k, B) \geqslant \frac{1}{n} > 0$ which is excluded. Hence $d(A \cap B, B \cap K_n)$ must be positive. In addition, since $A_n \subseteq A$ we have that $\mu^*(A) \geqslant \mu^*(A_n)$ for all $n \in \mathbb{N}$. Since $\{A \cap K_n\}$ is non decreasing, we have[1] $\lim_{n\to\infty} \mu^*(A \cap K_n) = \mu^*\left(\bigcup_{n\geqslant 1} A \cap K_n\right) = \mu(A \cap \{x \in \mathbb{R} : d(x,B) > 0\}) = \mu^*\left(A \cap \overline{B}'\right) = \mu^*(A \cap B')$ because B is closed. Then $\mu^*(A) \geqslant \lim_{n\to\infty} \mu^*(A_n) = \mu^*(A \cap B) + \lim_{n\to\infty} \mu^*(A \cap K_n) = \mu^*(A \cap B) + \mu^*(A \cap B')$. Therefore any closed subset of $\mathbb{R}$ is μ^*-measurable.

Next consider $\mathcal{M} := \{E \subset \mathbb{R} : \text{E is } \mu^* - \text{measurable}\}$. Plainly $\mathcal{M} \neq \emptyset$ since it contains all the closed subsets of $\mathbb{R}$. In particular $\emptyset, \mathbb{R} \in \mathcal{M}$. Let $E \in \mathcal{M}$ be given. For all $A \subseteq \mathbb{R}$, we have that $\mu^*(A) = \mu(A \cap E) + \mu(A \cap E') = \mu(A \cap E') + \mu\left(A \cap (E')'\right)$ which implies $E' \in \mathcal{M}$. Finally, let $\{E_k\}_{k\geqslant 1}$ be a countable family of pairwise disjoint μ^*-measurable of sets. We have for all $A \subseteq \mathbb{R}$ ($\sigma-$subadditivity)

$$\mu^*(A) \leqslant \mu^*\left(A \cap \left(\bigcup_{n\geqslant 1} E_n\right)\right) + \mu^*\left(A \cap \left(\bigcup_{n\geqslant 1} E_n\right)'\right) \leqslant \sum_{n=1}^{\infty} \mu^*(A \cap E_n) + \mu^*\left(A \cap \left(\bigcup_{n\geqslant 1} E_n\right)'\right).$$

We prove by induction on p that

$$\mu(A) = \sum_{n=1}^{p} \mu^*(A \cap E_n) + \mu^*\left(A \cap \left(\bigcup_{1\leqslant n\leqslant n} E_n\right)'\right). \qquad (*)$$

For $p = 1$, we have $\mu(A) = \mu^*(A \cap E_1) + \mu^*(A \cap E_1')$ which is true since E_1 is μ^*-measubrable. Suppose $(*)$ is verified at a step p. Since E_{p+1} is μ^*-measurable, we have

[1] This is because μ^* is continuous from below for increasing sequences (a property of outer measures).

$$\begin{aligned}\mu^*(A) &= \mu^*(A\cap E_{p+1}) + \mu^*\left(A\cap E'_{n+1}\right)\\ &= \mu^*(A\cap E_{p+1}) + \sum_{n=1}^{p}\mu^*(A\cap E_{p+1}\cap E_n) + \mu^*\left(A\cap E'_{p+1}\cap\left(\bigcup_{n=1}^{p}E_n\right)'\right)\\ &= \sum_{n=1}^{n+1}\mu^*(A\cap E_n) + \mu^*\left(A\cap\left(\bigcup_{n=1}^{p+1}E_n\right)'\right).\end{aligned}$$

Additionally, the sequence $\mu^*\left(\{\bigcup_{n=1}^{p}E_n\}'\right)$ is non increasing and bounded below by the number $\mu^*\left(A\cap(\bigcup_{n=1}^{\infty}E_n)'\right)$. Taking limits, we obtain

$$\mu^*(A) \geqslant \sum_{n=1}^{\infty}\mu^*(A\cap E_n) + \mu^*\left(A\cap\left(\bigcup_{n=1}^{\infty}E_n\right)'\right).$$

Thus $\mu^*(A) = \sum_{n=1}^{\infty}\mu^*(A\cap E_n) + \mu^*\left(A\cap(\bigcup_{n=1}^{\infty}E_n)'\right)$. Hence $\mathcal{M}$ is a $\sigma-$algebra containing the closed sets. Therefore $\mathcal{B}(\mathbb{R}) \subseteq \mathcal{M}$.