Classification of covering maps over paracompact spaces

Fabrice Razafimahatratra, Elmar Lasa

Clemson University

April 22, 2026

1. The classification problem
2. Coverings via configurations
3. Concluding remarks

1. The classification problem
2. Coverings via configurations
3. Concluding remarks

Coverings via configurations

For any space X, define the configuration space of ordered n-tuples of distinct points by

$$F_n(X) = \{(x_1, \ldots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}$$

- The symmetric group Σ_n acts freely on $F_n(X)$ by permuting coordinates.
- The quotient $F_n(X)/\Sigma_n$ is the space of *unordered* n-point subsets of X.
- Define

$$E_n(X) = \{(C, x) \in F_n(X)/\Sigma_n \times X \mid x \in C\}, \qquad \pi_n(C, x) = C.$$

Gauss maps

Definition

Let $p : E \to B$ be an n-fold covering map. A Gauss map for p is a map $g : E \longrightarrow \mathbb{R}^\infty$ such that for every $x \in B$ the restriction $g\big|_{p^{-1}(x)} : p^{-1}(x) \longrightarrow \mathbb{R}^\infty$ is injective.

- Each fiber $p^{-1}(x)$ has n points.
- A Gauss map assigns *distinct labels in* $\mathbb{R}^\infty$ to those n points.
- Continuity forces these labels to vary coherently from fiber to fiber.

Intuition

A Gauss map turns each fiber into a configuration of n distinct points of $\mathbb{R}^\infty$. That is the bridge from a concrete covering to the universal configuration-space cover.

Gauss map

Theorem

Let $p : E \to B$ be an n-fold covering map. Then the following are equivalent:

1. There exists a Gauss map $g : E \to \mathbb{R}^\infty$ for p.
2. There exists a map

$$f : B \to F_n(\mathbb{R}^\infty)/\Sigma_n$$

such that E is equivalent to the pullback covering $f^*E_n(\mathbb{R}^\infty)$.

Meaning

To classify an n-fold covering it is enough to package each fiber as an unordered configuration in $\mathbb{R}^\infty$.

Facts

Fact 1: Let $p' : E' \to B'$ be a locally trivial fiber bundle. If B is a *paracompact* space and $f, g : B \to B'$ are homotopic maps, then the pullback bundles f^*E' and g^*E' are isomorphic as bundles over B. That is,

$$f^*E' \cong g^*E' \qquad \text{(fiber bundles over X)}$$

Fact 2: Every n-fold covering map over a paracompact space has a Gauss map.

Lemma 1

Let $p : E \to B$ be an n-fold covering map, $1 \leq k \leq \infty$. There is a one-to-one correspondence between the set of bundle morphisms

$$\begin{tikzcd}
E \arrow[r, "\bar{f}"] \arrow[d] & E_n\left(\mathbb{R}^k\right) \arrow[d] \\
B \arrow[r, "f"'] & F_n\left(\mathbb{R}^k\right)/\Sigma_n
\end{tikzcd}$$

and the set of Gauss maps $g : E \longrightarrow \mathbb{R}^k$.

Lemma 2

Let $1 \leq k \leq \infty$. Suppose $G : E \times I \to \mathbb{R}^k$ is a homotopy trhough Gauss map, i.e., G_t is a Gauss map for all $t \in I$. Then, we obtain the bundle homomorphism

$$\begin{tikzcd}
E \times I \arrow[r, "\overline{F}"] \arrow[d, "p\times\mathrm{Id}"'] & E_n\left(\mathbb{R}^k\right) \arrow[d] \\
B \times I \arrow[r, "F"'] & F_n\left(\mathbb{R}^k\right)/\Sigma_n
\end{tikzcd}$$

with the property that, if $f_s : B \to F_n\left(\mathbb{R}^k\right)/\Sigma_n$, $s = 0, 1$ are the functions associated to G_s, then $F : f_0 \cong f_1$.

Classification result

Theorem

Let X be a paracompact space. Then there is a bijection

$$[X, F_n(\mathbb{R}^\infty)/\Sigma_n] \longrightarrow \mathcal{C}_n(X)$$

given by $[f] \longmapsto [f^* E_n(\mathbb{R}^\infty)]$.

- $\mathcal{C}_n(X)$ denotes the class of equivalence classes of all n-fold covering maps over X.
- $[X, F_n(\mathbb{R}^\infty)/\Sigma_n]$ represents the set of homotopy classes from X to the (unordered) configuration space $F_n(\mathbb{R}^\infty)/\Sigma_n$. Its elements are commonly known as *classifying maps*.
- $E_n(\mathbb{R}^\infty)$ is the total space associated with the n-fold covering $\pi_n : E_n(\mathbb{R}^\infty) \longrightarrow F_n(\mathbb{R}^\infty)/\Sigma_n$.

Proof of the classification result

Define $\Phi : \; [f] \longmapsto [f^* E_n(\mathbb{R}^\infty)]$.

Step 1

We first prove the map Φ is well-defined.

- If $f \simeq g$ are in the same homotopy class, then according to *fact 1*, $f^* E_n(\mathbb{R}^\infty) \cong g^* E_n(\mathbb{R}^\infty)$.
- Thus Φ is well-defined.

Step 2

We prove Φ is surjective

- Any n-fold covering $p : E_n(X) \longrightarrow X$ is associated with a Gauss map $g : E_n(X) \longrightarrow \mathbb{R}^\infty$.
- In turn the latter is associated with a classifying map $f_n : X \longrightarrow F_n(\mathbb{R}^\infty)/\Sigma_n$ such that the pullback covering satisfies $f^* E_n(\mathbb{R}^\infty) \cong E_n(X)$.

Proof of the classification result

Step 3

The next step is to prove the injectivity of Φ.

- We have the direct sum $\mathbb{R}^{\infty} = \mathbb{R}^{\infty}_{1} \oplus \mathbb{R}^{\infty}_{2}$ where $\mathbb{R}^{\infty}_{1} = \{\{t_i\}_{i=1}^{\infty} : t_{2i} = 0,\ i = 1, 2, 3, \dots\}$ and $\mathbb{R}^{\infty}_{1} = \{\{t_i\}_{i=1}^{\infty} : t_{2i-1} = 0,\ i = 1, 2, 3, \dots\}$.
- Consider two homotopies $h^1 : ((t_i), t) \mapsto (1-t)(t_i) + t(t_1, 0, t_2, 0, t_3, \dots)$ and $h^2 : ((t_i), t) \mapsto (1-t)(t_i) + t(0, t_1, 0, t_2, 0, t_3, \dots)$, with $(t_i) \in \mathbb{R}^{\infty}$ and $t \in I$.
- Call h^{ν}_1 their ‘ending maps’, $\nu = 1, 2$.
- $h^{\nu}_1 \circ p_2 : E_n(\mathbb{R}^{\infty}) \longrightarrow \mathbb{R}^{\infty}$ for $\nu = 1, 2$ are Gauss maps, with $p_2(C, x) = x$.

Proof of the classification result

- As a result, there are two bundle morphisms ($\nu = 1, 2$):

$$\begin{tikzcd}
E_n\left(\mathbb{R}^\infty\right) \arrow[r, "\overline{\varphi_\nu}"] \arrow[d, "h_1^\nu \circ p_2"'] & E_n\left(\mathbb{R}^\infty\right) \arrow[d] \\
\mathbb{R}^\infty \arrow[r, "\varphi_\nu"'] & F_n\left(\mathbb{R}^\infty\right)/\Sigma_n
\end{tikzcd}$$

- In the same vein $h^\nu \circ (p_2 \times \mathrm{Id}) := \; E_n\left(\mathbb{R}^\infty\right) \times I \longrightarrow \mathbb{R}^\infty$, $\nu = 1, 2$ are two homotopies whose restrictions $G_{t,\nu} : h^\nu \circ (p_2 \times \mathrm{Id})\big|_{E_n(\mathbb{R}^\infty)\times\{t\}}$ are Gauss maps, for all $t \in I$.

Proof of the classification result

- We check that $G_{0,\nu} \cong p_2$ and $G_{1,\nu} \cong h_1^{\nu} \circ p_2$.
- Thus, $G_{1,\nu}$ induces φ_ν which must be homotopic to the map induced by $p_2 \cong G_{0,\nu}$. But the map, say ϕ, induced by p_2 has to be homotopic to the identity as a consequence of the relation $\phi \circ p_2 = \pi_n \circ \overline{\phi}$.
- Now, assume $f_\nu : X \longrightarrow F_n(\mathbb{R}^\infty)/\Sigma_n$, $\nu = 1, 2$ are given such that $f_1^* E_n(\mathbb{R}^\infty) \cong f_2^* E_n(\mathbb{R}^\infty)$.
- Since both are isomorphic, the two bundles have homeomorphic total space and we have two morphisms of covering maps

$$\begin{tikzcd}
E \arrow[r, "\overline{f_\nu}"] \arrow[d] & E_n(\mathbb{R}^\infty) \arrow[d] \\
X \arrow[r, "f_\nu"'] & F_n(\mathbb{R}^\infty)/\Sigma_n
\end{tikzcd}$$

where $E := f_1^* E_n(\mathbb{R}^\infty)$.

Proof of the classification result

- The use of the associated Gauss map g_ν, $\nu = 1, 2$ allows the construction of the composite maps $h_1^\nu \circ g_\nu : E \longrightarrow \mathbb{R}^\infty$ which are again Gauss maps. Then we have

$$\begin{tikzcd}
E \arrow[r, "\overline{f_\nu}"] \arrow[d, "h_1^\nu \circ g_\nu"'] & E_n\left(\mathbb{R}^\infty\right) \arrow[r, "\overline{\varphi_\nu}"] \arrow[d] & E_n\left(\mathbb{R}^\infty\right) \arrow[d] \\
\mathbb{R}^\infty \arrow[r, "f_\nu"'] & F_n\left(\mathbb{R}^\infty\right)/\Sigma_n \arrow[r, "\varphi_\nu"'] & F_n\left(\mathbb{R}^\infty\right)/\Sigma_n
\end{tikzcd} \quad , \qquad \nu = 1, 2.$$

- Lastly, using the homotopy $G : E \times I \longrightarrow \mathbb{R}^\infty$ defined by $G(e,t) = (1-t)\, h_1^1 g_1(e) + t h_1^2 g_2(e)$ and *lemma 2*, we have $\varphi_1 \circ f_1 \simeq \varphi_2 \circ f_2$. The result follows since $\varphi_\nu \simeq \mathrm{Id}$ for $\nu = 1, 2$. □

1. The classification problem

2. Coverings via configurations

3. Concluding remarks

Concluding remarks

- Gauss maps turn fibers into configurations in $\mathbb{R}^\infty$.
- Theorem 1 converts that geometric data into a classifying map.
- Over paracompact spaces, every finite covering has such a Gauss map.
- Therefore $F_n(\mathbb{R}^\infty)/\Sigma_n \simeq B\Sigma_n$ classifies n-fold coverings.